Brief Report

The Evolution With Age of Probabilistic, Intuitively Based Misconceptions

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The purpose of this research was to investigate the evolution, with age, of probabilistic, intuitively based misconceptions. We hypothesized, on the basis of previous research with infinity concepts, that these misconceptions would stabilize during the emergence of the formal operation period. The responses to probability problems of students in Grades 5, 7, 9, and 11 and of prospective teachers indicated, contrary to our hypothesis, that some misconceptions grew stronger with age, whereas others grew weaker. Only one misconception investigated was stable across ages. An attempt was made to find a theoretical explanation for this rather strange and complex situation.

In a previous study we found that various intuitively based misconceptions related to the notion of infinity were relatively stable across ages, beginning at the formal operational period (Fischbein, Tirosh, & Hess, 1979). The main purpose of the present study was to investigate whether this finding is generally true by extending our research to a different domain, namely that of probabilistic intuitions. As far as we know, the evolution of probabilistic intuitions with age has not been extensively studied. (For research on the developmental aspects of probabilistic intuitions see Fischbein, 1975; Fischbein & Gazit, 1984; Fischbein, Nello, & Marino, 1991; Garfield & Ahlgren, 1988; Green, 1983; Hawkins & Kapadia, 1984; Piaget & Inhelder, 1951; and Shaughnessy, 1992.) In addition, we hoped this study would lead to a deeper understanding of the mechanisms that contribute to intuitive misconceptions in general and probabilistic misconceptions in particular.

The Concept of Intuition

We have previously defined the concept of intuition as a cognition that appears *subjectively* as self-evident, directly acceptable, holistic, coercive, and extrapolative (Fischbein, 1987). An intuitive cognition is distinguished from an analytically and logically based cognition by the feeling of obviousness, of intrinsic certainty. For example, I am sure that the sum of the angles of a triangle is 180° because I have been taught this or because I can prove it. But it is not obvious that it must be so. On the other hand, the fact that the shortest distance between two points is a straight line subjectively appears to be absolutely true without the need for any formal or empirical proof. In the first case we deal with a nonintuitive cognition, and in the second with an intuitive cognition.

Why do some of our cognitions have the character of evidence and direct imperativeness while others do not? We hypothesize that the reason is that we naturally tend to organize and to integrate our cognitions into coherent and behaviorally efficient structures. The result is the development with age and experience of crystallized, firm, and stable beliefs on which we may rely in our mental and practical behavior. These cognitive beliefs may, however, conflict with reality. One reason for this conflict is that our experience is usually limited. Another is that the need to act purposefully implies that we may manipulate causes to get a desired effect. As a consequence we tend, in interpreting events, to distinguish between cause and effect, and we tend to believe that a certain cause will always produce the same effect. Although such beliefs may help in the elaboration of internally coherent representations, these representations may be in dissonance with reality.

In seeking coherence for our cognitive organization, we tend in the course of our mental operations to integrate information that is easily available and to ignore information that requires a more sophisticated research effort. Also, we tend to rely on information that seems to be representative for an entire class. Finally—and this is a fundamental aspect of the entire theory of intuitive knowledge—the aspects and conditions mentioned above cannot work independently of what Piaget (1976) called the operational (logical, analytical) capacities of the individual. These additional factors caused us to extend our original research question: How do the factors related to coherence and efficiency influence the development of intuitions with age? It would appear that the impact of logical constraints on intuitions might increase with age; that is, if intuitions evolve with age, one might expect that the strength and frequency of intuitively based misconceptions would diminish as the subject grows older. But this logical conclusion contradicts our hypothesis, derived from previous findings concerning intuitions related to infinity, that intuitions are stable across ages. As we shall see, our initial hypothesis was found to be too simplistic when compared with the reality of experimental findings in the domain of probability.

METHODOLOGY

Subjects

Five groups of students were investigated: 20 students in Grade 5 (ages 10–11), 20 students in Grade 7 (ages 12–13), 20 students in Grade 9 (ages 14–15), 20 students in Grade 11 (ages 16–17), and 18 college students who were prospective teachers specializing in mathematics. The 11th graders were in the average ability level of the three levels of instruction in Israeli high schools. None of the students had previously received any instruction in probability. The students received no information regarding the purpose of the study. The sample represented a range of students with respect to socioeconomic level and cultural background.

Instrument

A questionnaire consisting of seven probability problems was developed. Each problem was related to a well-known probabilistic misconception (see the first column of Table 1). Answers were to be written. The questionnaire was administered to each group of students during a regularly scheduled class, under usual classroom conditions, in a session lasting about one hour.

Table 1
Problems and percentages of student answers.

Problems	Grades				
	5	7	9	11	CS[a]
1. Representativeness					
In a lotto game, one has to choose 6 numbers from a total of 40. Vered has chosen 1, 2, 3, 4, 5, 6. Ruth has chosen 39, 1, 17, 33, 8, 27. Who has a greater chance of winning?					
Vered has a greater chance of winning.	0	0	0	0	0
Ruth has a greater chance of winning. (Main misconception)	**70**	**55**	**35**	**35**	**22**
Vered and Ruth have the same chance to win. (Correct)	30	45	65	65	78
2. Negative and Positive Recency Effects					
When tossing a coin, there are two possible outcomes: either heads or tails. Ronni flipped a coin three times and in all cases heads came up. Ronni intends to flip the coin again. What is the chance of getting heads the fourth time?					
Smaller than the chance of getting tails (Main misconception; negative recency effect.)	**35**	**35**	**20**	**10**	**0**
Equal to the chance of getting tails (Correct.)	40	55	70	90	94
Greater than the chance of getting tails (Positive recency effect.)	0	5	0	0	6
Other types of answers	25	5	10	0	0
3. Compound and Simple Events					
Suppose one rolls two dice simultaneously. Which of the following has a greater chance of happening?					
Getting the pair 5-6 (Correct)	15	20	10	25	6
Getting the pair 6-6	0	0	0	0	0
Both have the same chance. (Main misconception)	**70**	**70**	**75**	**75**	**78**
Other answers	15	10	15	0	16
4. The Conjunction Fallacy					
Dan dreams of becoming a doctor. He likes to help people. When he was in high school, he volunteered for the Red Cross organization. He accomplished his studies with high performance and served in the army as a medical attendant. After ending his army service, Dan registered at the university. Which seems to you to be more likely?					
Dan is a student of the medical school. (Misconception)	**85**	**70**	**80**	**40**	**44**
Dan is a student.	15	30	20	60	56
5A. Effect of Sample Size					
In a certain town there are two hospitals, a small one in which there are, on the average, about 15 births a day and a big one in which there are, on the average, about 45 births a day. The likelihood of giving birth to a boy is about 50%. (Nevertheless, there were days on which more than 50% of the babies born were boys, and there were days on which fewer than 50% of the babies born were boys.) In the small					

(table continued)

Table 1—continued
Problems and percentages of student answers.

Problems	Grades 5	7	9	11	CS[a]
5A. Effect of Sample Size—continued					
hospital a record has been kept during the year of the days in which the total number of boys born was greater than 9, which represents more than 60% of the total births in the small hospital. In the big hospital, they have kept a record during the year of the days in which there were more than 27 boys born, which represents more than 60% of the births. In which of the two hospitals were there more such days?					
In the big hospital there were more days recorded where more than 60% boys were born.	20	35	5	10	0
In the small hospital there were more days recorded where more than 60% boys were born. (Correct)	0	0	5	0	0
The number of days for which more than 60% boys were born was equal in the two hospitals. (Main misconception)	**10**	**30**	**70**	**80**	**89**
Other answers	10	5	5	10	0
No answer	60	30	15	0	11
5B. The Effect of Sample Size					
The likelihood of getting heads at least twice when tossing three coins is:					
Smaller than (Incorrect)	5	5	25	10	6
Equal to (Incorrect; main misconception)	**30**	**45**	**60**	**75**	**44**
Greater than (Correct)	35	30	10	5	50
the likelihood of getting heads at least 200 times out of 300 times.					
Other answers	5	10	0	0	0
No answer	25	10	5	10	0
6. The Heuristic of Availability					
When choosing a committee composed of 2 members from among 10 candidates the number of possibilities is					
Smaller than (Incorrect)	20	5	10	0	22
Equal to (Correct)	0	5	5	15	6
Greater than (Main misconception)	**10**	**20**	**65**	**85**	**72**
the number of possibilities when choosing a committee of 8 members from among 10 candidates.					
Other answers	15	30	15	0	0
No answer	55	40	5	0	0
7. The Effect of the Time Axis (The Falk Phenomenon)					
Yoav and Galit each receive a box containing two white marbles and two black marbles.					
(A.) Yoav extracts a marble from his box and finds out that it is a white one. Without replacing the first marble, he extracts a second marble. Is the likelihood that this second marble is also white smaller than, equal to, or greater than the likelihood that it is a black marble?					
(B.) Galit extracts a marble from her box and puts it aside without looking at it. She then extracts a second marble and sees that it is white. Is the likelihood that the first marble she extracted is white smaller than, equal to, or greater than the likelihood that it is black?					

(table continued)

Table 1—continued
Problems and percentages of student answers.

Problems	Grades				
	5	7	9	11	CS[a]
Category 1 (Both correct)	45	50	35	30	39
Category 2 (First correct, second incorrect. Main misconception)	**5**	**30**	**35**	**70**	**44**
Category 3 (Both incorrect: Equal chances for both.)	25	15	25	0	0
Others	25	5	5	0	17

[a]College students.
[b]Main misconception responses are highlighted.

Problem 1 tested for the misconception of *representativeness*. People tend to estimate the likelihood of an event by taking into account how well it represents some aspect of its parent population (see Kahneman & Tversky, 1972; Shaughnessy, 1992; Tversky & Kahneman, 1982).

Problem 2 tested for *negative and positive recency effects*. One who tosses a coin three times and gets three heads may then believe that the fourth toss is more likely to be tails. This is called "the negative recency effect" or "the gambler's fallacy." This belief may be related to the heuristic of representativeness: Intuitively, alternating outcomes seem to better represent a random sequence; however, the belief that the fourth toss is more likely to be heads on the basis, for example, of an implicit or explicit assumption that the conditions were not fair is called "the positive recency effect" (see Cohen, 1957; Fischbein, 1975; Fischbein, Nello, & Marino, 1991).

Problem 3 tested for *simple and compound events*. For example, if two dice are rolled simultaneously, the tendency is to say that obtaining two sixes has the same likelihood as obtaining a five and a six (see Lecoutre & Durant, 1988).

Problem 4 tested for *the conjunction fallacy*. The probability of an event appears, under certain conditions, to be higher than the probability of the intersection of the same event with another (see Shaughnessy, 1992; Tversky & Kahneman, 1983).

Problems 5A and 5B tested for *the effect of sample size*. Individuals tend to neglect the influence of the magnitude of a sample when estimating probabilities (see Tversky & Kahneman, 1982).

Problem 6 tested for *availability*. Frequency or probability is estimated by the ease with which instances can be brought to mind (see Kahneman & Tversky, 1973).

Problem 7 tested for *the time-axis fallacy* (also called *the Falk phenomenon*). In this problem people are likely to answer Part 1 correctly, then answer Part 2 differently on the basis of the principle that an event cannot act retroactively on its cause. An inversion of the time axis, of cause implying effect, contradicts one of our basic intuitions (see Falk, 1979, 1983; Shaughnessy, 1992).

Remarks Concerning the Methodology

The present research constitutes the first stage of a larger project. The purpose here was to gather preliminary empirical data that would help us obtain a global picture of the evolution of probabilistic misconceptions as an effect of age. We intend in the second stage of the project to undertake a number of interviews based on the findings discussed here. Because of the lack of literature on the evolution of probability concepts with age, the basic theoretical ingredients for analyzing the mechanisms of this evolution were not available to us.

RESULTS

Although we collected examples of student justifications of answers, we will not include them here. The problem of the relationship between the initial direct answer and its justification is a complex one (Konold, Pollatsek, Well, Lohmeier, & Lipson, 1993). The initial answer expresses immediate intuition, while the justification, coming after the expressed intuition, may or may not reflect the rationale for the subject's choice; it may be a subsequent logical construction. Consequently, at this stage of our research we will not try to interpret the subjects' reactions in terms of their own justifications. In the next stage an attempt will be made to cope with this complex theoretical and methodological problem.

We next consider the results pertaining to each misconception, as shown in Table 1.

Representativeness. The representativeness misconception decreased with age.

Negative and positive recency effects. The impact of the negative recency strategy decreased with age, whereas the positive recency effect was almost absent. Konold et al. (1993) have shown that some subjects believe that they are asked to predict a certain outcome, whereas others consider that they have to evaluate the probability of a string of outcomes. Between the two approaches there may arise conflicts that lead to inconsistencies, even for the same subject.

Compound and simple events. This misconception was frequent and stable across ages, the only stable misconception identified in this study.

The conjunction fallacy. This misconception was very strong through Grade 9 but less strong (by about half) for high school and college students.

Effect of sample size. For Problems 5A and 5B the basic misconception is that sample size is not relevant. This misconception is related to the heuristic of representativeness (Tversky & Kahneman, 1982). Table 1 shows that for 5A this misconception *developed* with age in a surprisingly regular manner, while the correct answer (the small hospital) was almost never chosen. For 5B the frequencies of the main misconception again increased with the age of the student, except in the case of college students. The basic belief expressed in this misconception is based on the idea that a ratio is representative of an indefinite number of pairs of numbers, a belief so strong that it masks a more subtle idea: As the considered sample becomes larger, the probability of getting a certain empirical result tends to better approximate the theoretical prediction. On the basis of this finding, we assume that a certain logical position that improves as the student's age increases (as a result of instruction and intellectual development) may become an influential component of an intuitive interpretation. The influence of such a position may be positive when it is adequate (for instance, in the example of the negative recency effect) but destructive when the logical view is inadequate, such as in the case of believing sample size to be irrelevant.

Availability. Once again we have a case in which the frequency of the misconception increased with the age of the student. The complementarity and subsequent equality of the two groups was not grasped intuitively. The only explanation we could find is that as subjects grow older, they become better able to identify possible combinations. But because it is easier to produce various combinations of two elements than combinations of eight elements (availability), *two elements* is selected as the answer.

The effect of the time axis (the Falk phenomenon). Our wording of this problem avoided the term *probability*. We divided the responses into three categories: In Category I both responses are correct; in Category II the first response is correct while the second is incorrect; and in Category III both responses are incorrect. Category II represents the main misconception. Here we have another example in which the frequency of an intuitively based misconception increased with age (except in the case of college students). We consider this response at two levels of rationalization. In this problem the general principle of causality, with its apparent one-directedness, seemed to become more effective in structuring the intuitive interpretation as the age of the subject increased: What happens at the second extraction cannot retroactively influence what has already happened in the first extraction. What the subjects did not seem to realize is that the *knowledge* of the second outcome should be used in determining the probability of the first outcome. If we know the second draw is white, then of the remaining three, from which the first one was drawn, two are black and one is white, so it is more likely that a black one was drawn. The apparently causal order of the story as it is told in a sequence of events hides the genuine stochastic structure of the problem: The two questions actually express the same problem. The erroneous intuition is caused by the tacit embedding of the principle of causality, with its unique time-direction, in the intuitive evaluation.

DISCUSSION AND CONCLUSIONS

Our purpose was to investigate the evolution, as students age, of probabilistic misconceptions. Our initial hypothesis was that after the emergence of formal reasoning (about age 12), intuitions tend to stabilize and become resistant to the influence of age and instruction. This hypothesis was suggested by our previous findings with regard to intuitions about infinity. In this first phase of our present study we selected a set of well-known probability problems that have been described as leading to intuitively based misconceptions.

The results obtained were contrary to the general assumption about the stability of intuitions. Only the problem on the simultaneous rolling of two dice yielded stable frequencies across ages. For the other six problems the frequencies varied across ages. In two cases (the problem on representativeness and the one on negative recency effect) the frequencies of the typical misconceptions diminished as the student aged. The misconception described as the *conjunction fallacy* was relatively constant in Grades 5, 7, and 9, then dropped for Grade 11 and for college students. In the other three cases (the problems on the effect of sample size on availability, and on the effect of the time axis) the frequencies of wrong intuitive answers increased with age, except in the case of college students, for whom, in most cases, frequencies of the misconceptions decreased. One may assume that the increased mathematical experience (and possibly the increased maturity of reasoning) of college students was strong enough to oppose the effect of the respective misconceptions. In the end then, the picture was rather complex: Some misconceptions diminished with age, one was stable, and some gained greater influence.

One plausible interpretation of these findings is that in each intuition there is embedded a certain intellectual schema that influenced the results. The schema acted tacitly and, in our opinion, became an integral part of the respective intuition. As the students aged, these intellectual schemata (general principles) became stronger and better integrated into the intellectual activity of the individual, and consequently were more influential in the individual's theoretical decisions.

Let us consider the principles that can be identified as having influenced the intuitions that guided the decisions made in these problems.

1. For the problems relating to representativeness (the negative recency effect and the higher likelihood of a group of random numbers winning in a lottery game), the basic principle is the independence of outcomes in a stochastic experience. It is this principle, this intellectual schema, that improves as the student ages and finally overcomes the primitive, global, intuitive heuristic of representativeness.

2. The general principle identifiable in the problems relating to the effect of sample size is the equivalence of ratios. For example, the concept of ratio is involved in the students' incorrect solution of the problem of the two hospitals. Students are apparently misled by their belief that one must use ratios to solve this problem. Instead, one has to consider another stochastic law, the law of large numbers. As the sample size (or the number of trials) increases, the relative frequencies tend toward the theoretical probabilities. For our subjects, not trained in stochastics, the principle of equivalence of ratios imposes itself as relevant to the problem and thus dictates the answer. It is the evolution of this principle that shapes the evolution of the related misconception and causes it to become stronger as the student ages.

3. The Falk phenomenon, the misconception found in responses to the last problem, is that the second extraction cannot influence the previous extraction. Here, too, there is deeply rooted in our mental activity a general principle that determines the answer. It is the principle of causality: The antecedent determines the consequent. The strength of this principle leads the individual to neglect an essential bit of information: The second marble extracted is known to be white. Here one must make the subtle shift from a concrete, causal, time-oriented relation to a formal mathematical relation.

Our findings show that with age, students become better able to employ the principles of causality and irreversibility of time, despite the fact that these principles alone are inadequate for solving the problem.

Generally speaking, in the students' intuitive solutions to these problems, we can identify the following common structure: The solution is shaped by the interaction between a general intellectual schema, accepted intuitively by the student, and certain specific constraints of the problem. The impact of these schemata increases with the age of the subject. The interventions of the schemata (tacit, of course) may be adequate or inadequate. When the constraints of the problem are simple enough so that the general principle is adequate to address them, frequencies of the respective misconceptions diminish as the student ages. In other situations the general schemata, though meaningful in themselves, are inadequate to deal with the specific constraints of the problem, and the frequencies of the respective misconceptions increase with the student's age.

We believe it could be useful to discuss, during instruction on probability, problems such as those we have analyzed here. There are many such problems in the literature. We suggest that it may be appropriate not only to present the problem and its correct solution but also to analyze psychologically the structure of the corresponding misconceptions. Probability does not consist of mere technical information and procedures leading to solutions. Rather, it requires a way of thinking that is genuinely different from that required by most school mathematics. In learning probability, students must create new intuitions. Instruction can lead students to actively experience the conflicts between their primary intuitive schemata and the particular types of reasoning specific to stochastic situations. If students can learn to analyze the causes of these conflicts and mistakes, they may be able to overcome them and attain a genuine probabilistic way of thinking.

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