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Detectable Properties and Spectral Quantum Logics

by

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1. Introduction

We introduce the notions of counter and property as derived concepts within the framework of quantum logics. In the course of our investigations we exhibit a rather important class of quantum logics or orthomodular posets through the requirement that every counter is the expectation functional of some (bounded) observable. These so-called spectral orthomodular posets have many interesting features also as far as the poset of detectable properties is concerned. This study sheds some light on the interrelatedness of the quantum logic and empirical logic approach [5, 6, 10, 15, 16] to Ludwig's approach [12, 13] to the foundations of quantum mechanics both at the mathematical as well as the conceptual level.

2. Preliminaries

We begin with a brief account of the definitions and basic facts pertaining to the subject matter.

A poset $(L, \leqslant)$ with a smallest (0) and a largest element (1) together with a map $p \in L \to p' \in L$ satisfying

(i) if $p \leqslant q'$ then $p \vee q$ exists,

(ii) $(p')' = p$,

(iii) $p \vee p' = 1$,

(iv) $p \leqslant q \Rightarrow q' \leqslant p'$,

(v) $p \leqslant q'$, $p \vee q = 1 \Rightarrow p = q'$

is called *orthomodular*; the elements are called *propositions*. A pair $p, q \in L$, $(L, \leqslant, ')$ an orthomodular poset, is said to be *orthogonal*, denoted $p \perp q$, provided $p \leqslant q'$. A subset $C \subseteq L - \{0\}$ is said to be *orthogonal* if $p, q \in C$ $p \neq q$ implies $p \perp q$. With this definition, $\emptyset \subseteq L$ and singletons $\{p\}$ $(p \neq 0)$ are orthogonal sets. A simple Zorn argument shows that every orthogonal set is contained in a maximal such set. Let $\mathcal{O}(L)$ denote the set of all maximal orthogonal sets and $\mathcal{O}_\sigma(L)$ the set of all maximal orthogonal sets with countably many elements (e.g.: $\{1\}$).

An element $\mu \in \mathbb{R}^L$ is called a *state on* L provided that

$$\mu(p \vee q) = \mu(p) + \mu(q) \text{ if } p \perp q,\ p, q \in L.$$

Note that the states on L form a vector subspace of $\mathbb{R}^L$. A state μ is called *positive* if $\mu(p) \geqslant 0$ for all $p \in L$. The set $K(L)$ of positive states is a cone in $\mathbb{R}^L$. A state μ is said to be *normalized* if $\mu(1) = 1$. A positive normalized state is called *probability state*. We note that the convex set $\Omega(L)$ of probability states is a base for the cone $K(L)$. We denote $V(L) := K(L) - K(L) = \text{lin}\ \Omega(L)$.

Let $E \in \mathcal{O}(L)$ and $\mu \in V(L)$ then the net $(\mu(VC))_{C \in E^f}$, where $(E^f, \subseteq)$ is the set of finite subsets of E directed by set-inclusion, converges in $\mathbb{R}$. Now, a state $\mu \in V(L)$ is called *completely additive*, resp. *σ-additive*, if for every $E \in \mathcal{O}(L)$, resp. $E \in \mathcal{O}_\sigma(L)$, $\mu(1) = \lim\ (\mu(VC))_{C \in E^f}$ holds true. The subspace of $V(L)$ of completely additive, resp. σ-additive, states is denoted by $U_c(L)$, resp. $U_\sigma(L)$. Also, we denote $\Omega_c(L) := U_c(L) \cap \Omega(L)$, $K_c(L) := U_c(L) \cap K(L)$, resp. $\Omega_\sigma(L) := U_\sigma(L) \cap \Omega(L)$, $K_\sigma(L) := U_\sigma(L) \cap K(L)$. Clearly, $K_c(L)$, resp. $K_\sigma(L)$, is a cone in $V(L)$ with $\Omega_c(L)$, resp. $\Omega_\sigma(L)$, as a base. We write $V_c(L) := K_c(L) - K_c(L) = \text{lin}\ \Omega_c(L)$ and $V_\sigma(L) := K_\sigma(L) - K_\sigma(L) = \text{lin}\ \Omega_\sigma(L)$ and remark that $(V(L), \Omega(L))$, resp. $(V_c(L), \Omega_c(L))$, resp. $(V_\sigma(L), \Omega_\sigma(L))$, is a base normed space provided, of course, that $\Omega(L)$, resp. $\Omega_c(L)$, resp. $\Omega_\sigma(L)$, is not empty. For the general theory of base normed spaces and order unit normed spaces see [1, 21] etc. and for a detailed discussion, in various contexts, of the spaces mentioned above refer to [2, 3, 4, 19].

In the sequel we shall be mainly concerned with the base normed space $(V_\sigma(L), \Omega_\sigma(L))$ and its Banachdual (with respect to the base norm) organized to an order unit normed space $(V_\sigma^*(L), \leqslant, e) : f \leqslant g :\Leftrightarrow f(\mu) \leqslant g(\mu)$ for all $\mu \in \Omega_\sigma(L)$, $f, g \in V_\sigma^*(L)$; e is the (unique) linear functional on $V_\sigma(L)$ with $e(\Omega_\sigma(L)) = 1$, an order unit of $(V_\sigma^*(L), \leqslant)$. We set $f' := e - f$ for $f \in V_\sigma^*(L)$. Note that $[0, e]$ $(= \{g \in V_\sigma^*(L) \mid 0 \leqslant g \leqslant e\})$ is closed under the map $f \to f'$. Also, $[-e, e]$ equals the unit ball in $V_\sigma^*(L)$.

With every proposition $p \in L$ we now associate an element of $V_\sigma^*(L)$ as follows: $f_p(\mu) := \mu(p)$. Clearly, $f_p \in [0, e]$ for all $p \in L$, furthermore

(i) $f_0 = 0$, $f_1 = e$,

(ii) $p \leqslant q \Rightarrow f_p \leqslant f_q$,

(iii) $p \perp q \Rightarrow f_{p \vee q} = f_p + f_q$,

(iv) $f_{p'} = (f_p)'$.

We denote $P := \{f_p \mid p \in L\}$.

As for the conceptual background, we may think of $(L, \leqslant, ')$ as the operational logic of a manual that is considered to be the appropriate "catalogue of questions" for a given empirical system. We also may assume that $\Omega_\sigma(L)$ consists exactly of the regular states, i.e. states induced by complete stochastic models of the manual [5, 6, 16].

3. Counters and Properties

An element $f \in [0, e]$ is called a *counter*. Note that the restriction of a counter to $\Omega_\sigma(L)$ is an affine functional taking on values between 0 and 1. Since every bounded affine functional on $\Omega_\sigma(L)$ admits a unique extension to a base norm continuous linear functional on $V_\sigma(L)$, we might just as well introduce counters as affine functionals on $\Omega_\sigma(L)$ bounded by 0 and 1.

The map $p \in L \to f_p \in [0, e]$ associates with each proposition a counter. We refer to the elements of P as *propositional counters*. Since P is total, i.e. $f(\mu) = 0$ for all $f \in P$ implies that $\mu = 0$, we conclude that $V_\sigma^*(L) = \sigma(V_\sigma^*(L), V_\sigma(L)) - \mathrm{cl}(\mathrm{lin}\, P)$ [11]. Therefore every counter is the weak*-limit of some net of linear combinations of propositional counters.

In this sense, counters are – at least in principle – operationally accessible by the apparatuses described by the manual. Having this in mind, we may interpret the value $f(\mu), f \in [0, e], \mu \in \Omega_\sigma(L)$ as the long-run relative frequency with which the counter f is triggered according to the σ-additive probability state (stochastic model) μ.

Following Mielnik [14], we define a *property* to be a subset $F \subseteq \Omega_\sigma(L)$ such that

for $\mu_1, \mu_2 \in \Omega_\sigma(L),\ t \in (0, 1)$: $t\mu_1 + (1-t)\mu_2 \in F \Leftrightarrow \mu_1, \mu_2 \in F$.

If $\mu \in \Omega_\sigma(L)$ is the σ-additive probability state describing a given system in the sense of a stochastic model for the manual and $\mu \in F$, F a property, then we say that *the system has property F*. With $(\mathcal{P}(L), \subseteq)$ we denote the set of properties ordered by set-inclusion, i.e. according to their generality. Clearly, $\emptyset \subseteq \Omega_\sigma(L)$ and $\Omega_\sigma(L)$ are properties, also the set-theoretical intersection of a family of properties is again a property. Therefore $(\mathcal{P}(L), \subseteq)$ is a complete lattice, the infimum of any family of properties being its set-theoretical intersection.

A pair (F, G) of properties is said to be *mutually exclusive*, denoted $F \dagger G$, provided there exists a counter $f \in [0, e]$ with $F \subseteq f^{-1}(1)$ and $G \subseteq f^{-1}(0)$. We say the counter f *separates* the pair of properties (F, G). Note, if (F, G) is separated by f then (G, F) is separated by f', thus the relation $\dagger$ is symmetric. Also, $F \dagger \emptyset$ for all $F \in \mathcal{P}(L)$, e.g. e separates the pair $(F, \emptyset)$. A pair of mutually exclusive properties may be separated by more that one counter.

One easily verifies that for a counter f, $f^{-1}(1) \cap \Omega_\sigma(L)$ and $f^{-1}(0) \cap \Omega_\sigma(L) = (f')^{-1}(1) \cap \Omega_\sigma(L)$ are properties. A property $F \in \mathcal{P}(L)$ is said to be *detectable* provided there exists a counter f such that $F = f^{-1}(1) \cap \Omega_\sigma(L)$; f is referred to as a counter *detecting* the property F. Here too, a detectable property may have more than one counter detecting it. The properties $\emptyset$ and $\Omega_\sigma(L)$ are detectable. With $\mathcal{D}(L)$ we denote the set of detectable properties.

Lemma 3.1: Let, $F, G \in \mathcal{D}(L)$ then $(\mathcal{D}(L), \subseteq)-\inf\{F, G\}$ exists in $(\mathcal{D}(L), \subseteq)$ and equals $F \cap G$.

Proof: There exist $f, g \in [0, e]$ with $F = f^{-1}(1) \cap \Omega_\sigma(L)$, $G = g^{-1}(1) \cap \Omega_\sigma(L)$. Then $1/2\, f + 1/2\, g \in [0, e]$ and therefore $(1/2\, f + 1/2\, g)^{-1}(1) \cap \Omega_\sigma(L)$ is a detectable property. If $1 = (1/2\, f + 1/2\, g)(\mu) = 1/2\, f(\mu) + 1/2\, g(\mu)$ for $\mu \in \Omega_\sigma(L)$ then $f(\mu) = g(\mu) = 1$, so $\mu \in F \cap G$. Conversely, if $\mu \in F \cap G$ then clearly $(1/2\, f + 1/2\, g)(\mu) = 1$. Hence, $(1/2\, f + 1/2\, g)^{-1}(1) \cap \Omega_\sigma(L) = F \cap G$.

A property F is said to be *semi-detectable* provided F is the set-theoretical intersection of some family of detectable properties. Let $S(L)$ be the set of semi-detectable properties; clearly $\mathcal{D}(L) \subseteq S(L)$ and the set-theoretical intersection of a family of semi-detectable properties is again semi-detectable. Note that $(P(L), \subseteq)$ and $(S(L), \subseteq)$ are complete lattices.

Quite often, propositions are considered as asserting "properties" of the system to be investigated. This can be made more precise as follows: if $p \in L$, we define $\hat{p} = \{\mu \in \Omega_\sigma(L) | \mu(p) = 1\}$, obviously a detectable property detected by the propositional counter f_p. A property F is said to be a *propositional property* provided there exists a proposition $p \in L$ with $\hat{p} = F$. The map $p \in L \to \hat{p} \in \mathcal{D}(L)$ satisfies

(i) $\hat{0} = \emptyset$, $\hat{1} = \Omega_\sigma(L)$,

(ii) $p \leqslant q \Rightarrow \hat{p} \subseteq \hat{q}$,

(iii) $p \perp q \Rightarrow \hat{p} \dagger \hat{q}$.

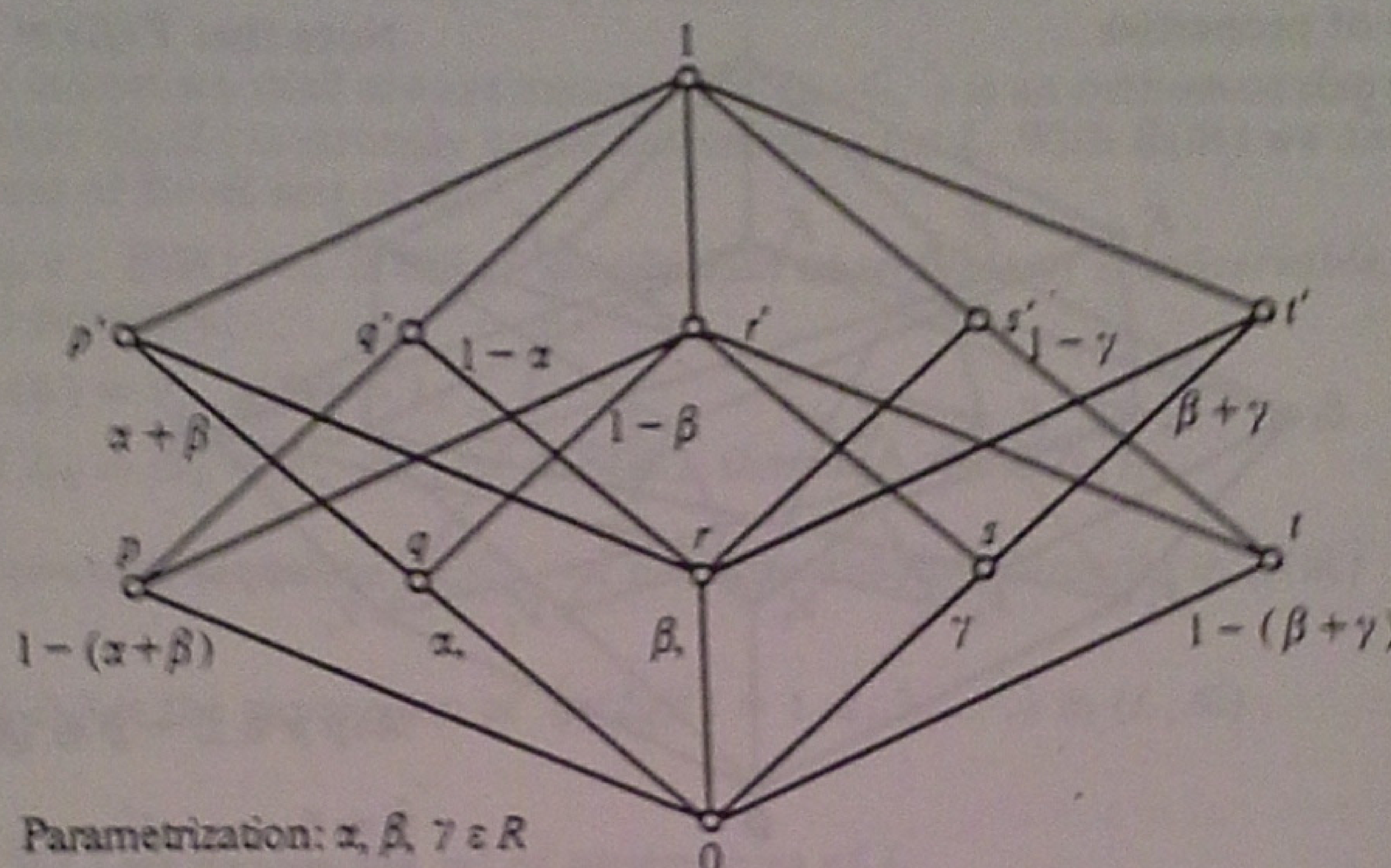
We say that $\Omega_\sigma(L)$ is *strongly order determining* for $(L, \leqslant, ')$ provided that $\hat{p} \subseteq \hat{q} \Rightarrow p \leqslant q$.

We now focus our interest on detectable properties, specifically on the structure $(\mathcal{D}(L), \subseteq, \dagger)$. Such structures arising from polytopes have been studied in their own right in [17] and measure-theoretic aspects are discussed in [20].

As an illustration let us consider the example outlined on pages 39 and 40.

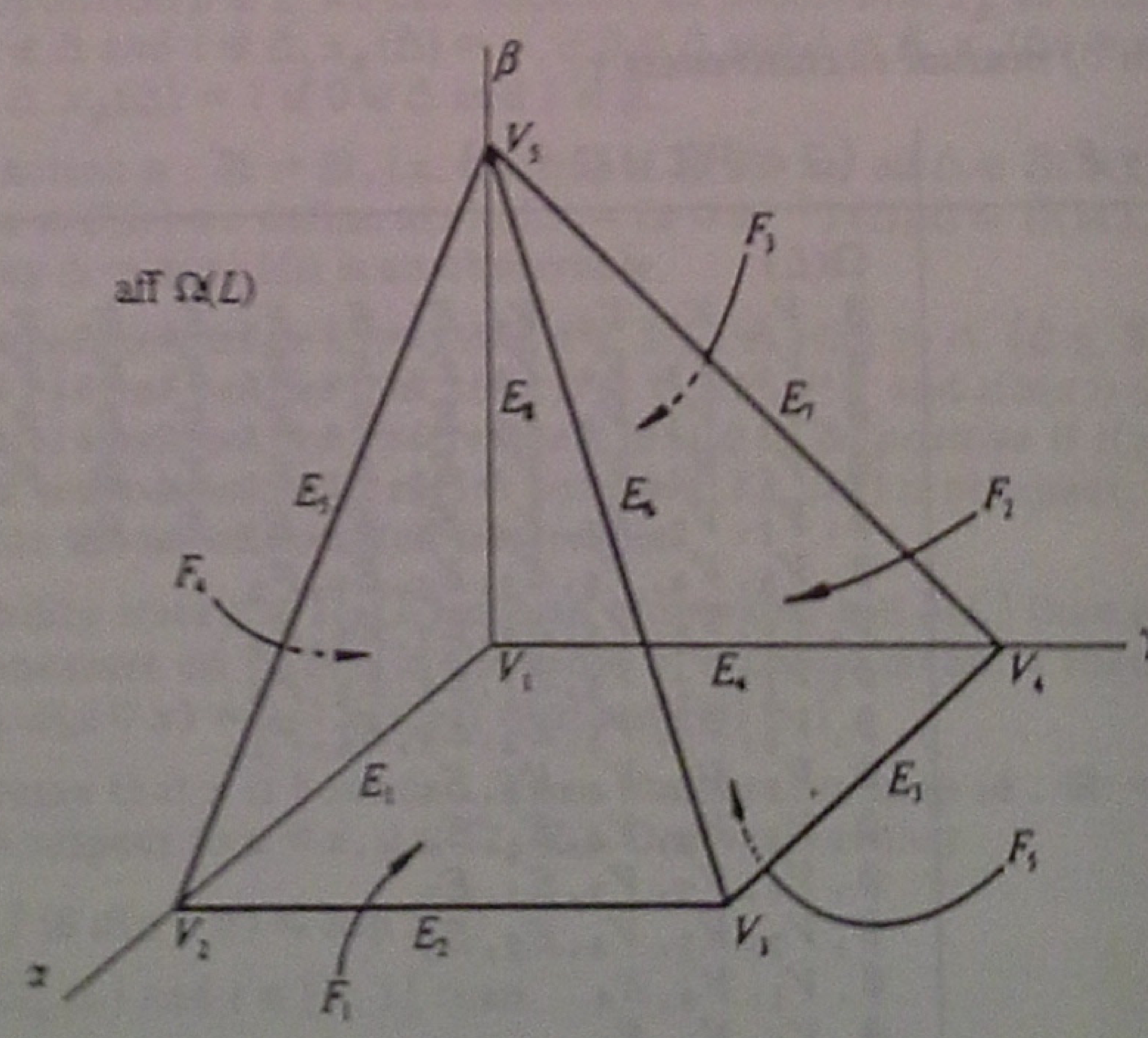
This example also shows that a detectable property need not be propositional. From a conceptual standpoint of view, however, it is highly desirable to know when the propositions match with the detectable properties, more precisely, when $(L, \leqslant, \perp)$ is isomorphic to $(\mathcal{D}(L), \leqslant, \dagger)$ under the map $p \to \hat{p}$.

Orthomodular poset $(L, \leq, ')$:



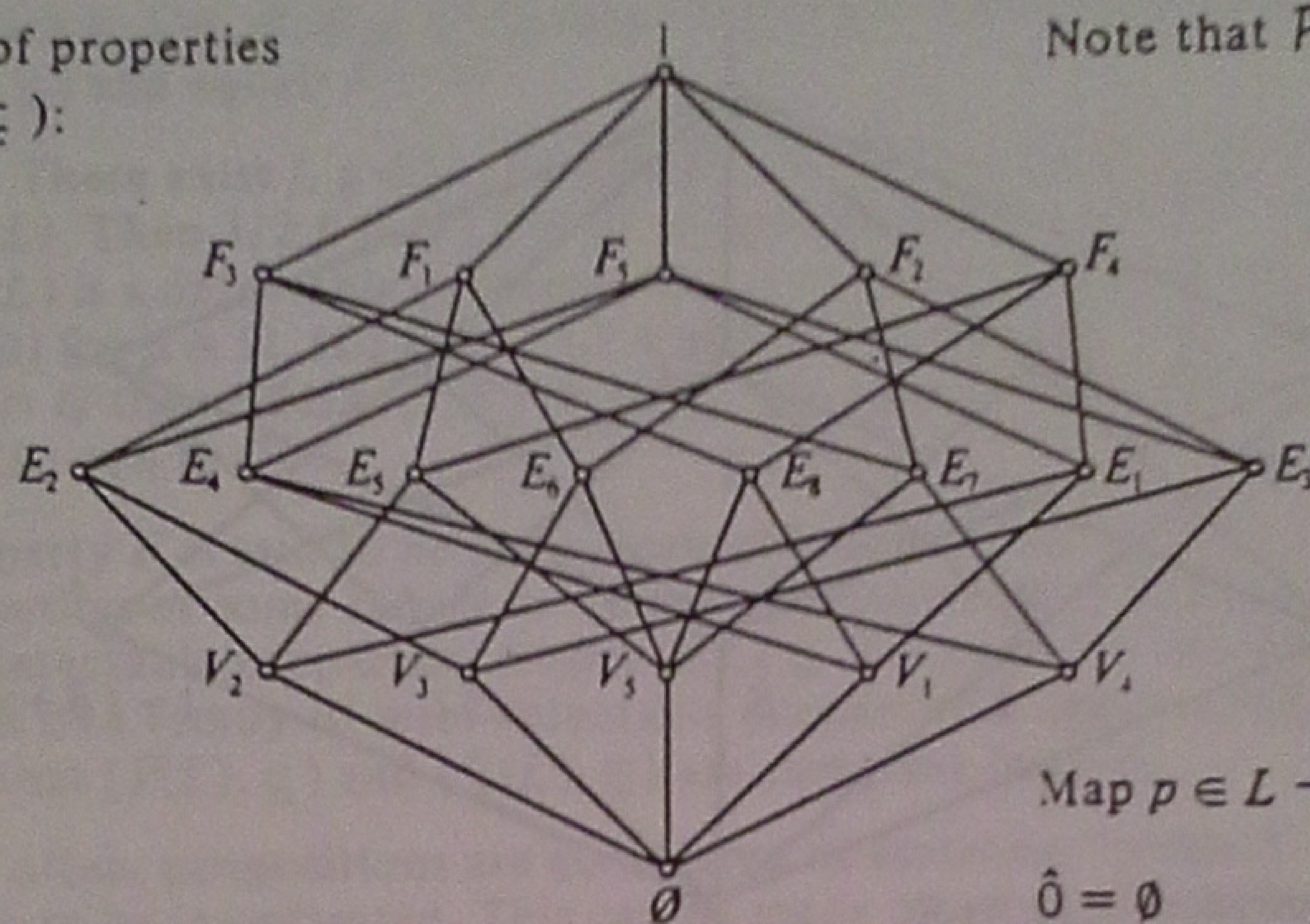
Parametrization: $\alpha, \beta, \gamma \in R$

Convex set of probability measures $\Omega(L) = \Omega_\sigma(L)$:



Note that $\Omega(L)$ is strongly order determining for L

Lattice of properties $(\mathcal{P}(L), \subseteq)$:



Note that $\mathcal{P}(L) = \mathcal{D}(L)$

Map $p \in L \mapsto \hat{p} \in \mathcal{D}(L)$:

$\hat{0} = \emptyset$	$\hat{p}' = F_1$
$\hat{r} = V_5$	$\hat{t}' = F_2$
$\hat{t} = E_1$	$\hat{q}' = F_3$
$\hat{q} = E_2$	$\hat{s}' = F_4$
$\hat{s} = E_3$	$\hat{r}' = F_5$
$\hat{p} = E_4$	$\hat{1} = \Omega(L)$

Relation of mutual exclusiveness $\dagger$:

Property F	$\{G \in \mathcal{P}(L) \mid G \dagger F\}$
$\emptyset$	$\Omega(L)$
V_1	$\emptyset, V_2, V_3, V_4, V_5, E_2, E_3, E_5, E_6, E_7, F_1, F_2$
V_2	$\emptyset, V_1, V_3, V_4, V_5, E_3, E_4, E_6, E_7, E_8, F_2, F_3$
V_3	$\emptyset, V_1, V_2, V_4, V_5, E_1, E_4, E_5, E_7, E_8, F_3, F_4$
V_4	$\emptyset, V_1, V_2, V_3, V_5, E_1, E_2, E_5, E_6, E_8, F_1, F_4$
V_5	$\emptyset, V_1, V_2, V_3, V_4, E_1, E_2, E_3, E_4, F_5$
E_1	$\emptyset, V_3, V_4, V_5, E_3, E_5, E_7, F_2$
E_2	$\emptyset, V_1, V_4, V_5, E_4, E_7, E_8, F_3$
E_3	$\emptyset, V_1, V_2, V_5, E_1, E_5, E_8, F_4$
E_4	$\emptyset, V_2, V_3, V_5, E_2, E_5, E_6, F_1$
E_5	$\emptyset, V_1, V_3, V_4, E_3, E_4$
E_6	$\emptyset, V_1, V_2, V_4, E_1, E_4$
E_7	$\emptyset, V_1, V_2, V_3, E_1, E_2$
E_8	$\emptyset, V_2, V_3, V_4, E_2, E_3$
F_1	$\emptyset, V_1, V_4, E_4$
F_2	$\emptyset, V_1, V_2, E_1$
F_3	$\emptyset, V_2, V_3, E_2$
F_4	$\emptyset, V_3, V_4, E_3$
F_5	$\emptyset, V_5$
$\Omega(L)$	$\emptyset$

4. Spectral Orthomodular Posets

In the sequel we shall always assume that $(L, \leq, ')$ is an orthomodular poset such that $\Omega_\sigma(L)$ is strongly order determining for L. With $\mathcal{B}(\mathbb{R})$ we denote the class of Borel sets of $\mathbb{R}$.

A map $x : \mathcal{B}(\mathbb{R}) \to L$ is called *Varadarajan observable of L* (*observable*, for short) provided:

(i) $x(\emptyset) = 0, \; x(\mathbb{R}) = 1$,

(ii) if $\Delta_1 \cap \Delta_2 = \emptyset, \; \Delta_1, \Delta_2 \in \mathcal{B}(\mathbb{R})$, then $x(\Delta_1) \perp x(\Delta_2)$,

(iii) for every sequence $(\Delta_i)_{i=1}^{\infty}$ of pairwise disjoint elements in $\mathcal{B}(\mathbb{R})$

$x(\bigcup_{i=1}^{\infty} \Delta_i)$ is the supremum of $\{x(\Delta_i) \mid i = 1, 2, 3, \ldots\}$ in $(L, \leq)$.

With $S(L)$ we denote the set of observables of L.

As the definition indicates, we do not assume that $(L, \leq, ')$ is σ-orthocomplete but only that the required supremum exists. We may consider the proposition $x(\Delta)$ as asserting that the "measurement" of the observable x yields a real number belonging to the Borel set Δ.

With every proposition $p \in L$ we can associate an observable x_p as follows: $x_p(\Delta) = 0$ if $0 \notin \Delta$ and $1 \notin \Delta$, $x_p(\Delta) = p'$ if $0 \in \Delta$ and $1 \notin \Delta$, $x_p(\Delta) = p$ if $0 \notin \Delta$ and $1 \in \Delta$, $x_p(\Delta) = 1$ if $0 \in \Delta$ and $1 \in \Delta$.

For a Borel function $\alpha : \mathbb{R} \to \mathbb{R}$, i.e. $\alpha^{-1}(\Delta) \in \mathcal{B}(\mathbb{R})$ for all $\Delta \in \mathcal{B}(\mathbb{R})$, and an observable $x \in S(L)$ we define $\alpha(x)(\Delta) := (x \circ \alpha^{-1})(\Delta)\; \Delta \in \mathcal{B}(\mathbb{R})$. Clearly, the map $\Delta \to \alpha(x)(\Delta)$ is an observable.

We define the *spectrum* of an observable $x \in S(L)$ as $s(x) := \cap \{\Delta \subseteq \mathbb{R} \mid \Delta$ closed, $x(\Delta) = 1\}$. One verifies that $x(s(x)) = 1, s(x) \neq \emptyset$ and $s(\alpha(x)) \subseteq cl - \alpha(s(x))$, α a Borel function. An observable x is said to be *positive* if $s(x) \subseteq [0, \infty)$, x is said to be *bounded* if $s(x)$ is bounded, i.e. $s(x)$ is compact. With $S^b(L)$ we denote the set of bounded observables.

Given a probability state $\mu \in \Omega_\sigma(L)$ and an observable $x \in S(L)$ then $\mu \circ x$ is probability measure on $\mathcal{B}(\mathbb{R})$ (in the sense of classical measure theory). Note that $\int \chi_R \, d(\mu \circ x) = \mu \circ x(s(x)) = \mu(x(s(x))) = 1$.

Let us now assume that x is bounded. Then the identity map id : $\mathbb{R} \to \mathbb{R}$ is integrable with respect to $\mu \circ x, \mu \in \Omega_\sigma(L)$. One verifies that

(i) $\inf s(x) \leq \int \mathrm{id} \, d(\mu \circ x) \leq \sup s(x)$,

(ii) inf $\mu, \nu \in \Omega_\sigma(L)$ and $t \in [0, 1]$ then

$\int \mathrm{id} \, d((t\mu + (1-t)\nu) \circ x) = \int \mathrm{id} \, d(t(\mu \circ x) + (1-t)(\nu \circ x)) =$
$= t \int \mathrm{id} \, d(\mu \circ x) + (1-t) \int \mathrm{id} \, d(\nu \circ x)$.

From this we conclude that the map $\mu \in \Omega_\sigma(L) \to \int \mathrm{id} \, d(\mu \circ x) \in R$ is a bounded affine functional on $\Omega_\sigma(L)$ and as such it has a unique extension to a base norm continuous linear functional on $V_\sigma(L)$, called the *expectation functional of* $x \in S^b(L)$ and denoted by $E(x)$. The value $E(x)(\mu)$ is called the *expectation of* x *in the state* $\mu \in V_\sigma(L)$. Also note that $\inf s(x)\, e \leqslant E(x) \leqslant \sup s(x)\, e$.

Lemma 4.1: Let $x \in S^b(L)$. Then $\inf s(x) = \inf\limits_{\mu \in \Omega_\sigma(L)} E(x)(\mu)$, $\sup s(x) = \sup\limits_{\mu \in \Omega_\sigma(L)} E(x)(\mu)$, $\|E(x)\| = \max\{\sup s(x), -\inf s(x)\}$.

Proof: We have $\inf s(x) \leqslant E(x)(\mu) \leqslant \sup s(x)$, $\mu \in \Omega_\sigma(L)$. Since $s(x)$ is closed, we conclude that $\inf s(x) \in s(x)$ thus

$$x(\inf s(x) - \epsilon;\ \sup s(x) + \epsilon) \neq 0, \qquad \epsilon > 0.$$

Therefore there exists $\mu \in \Omega_\sigma(L)$ such that $(\mu \circ x)(\inf s(x) - \epsilon; \sup s(x) + \epsilon) = 1$. Then

$$E(x)(\mu) = \int \chi_{(\inf s(x) + \epsilon;\, \inf s(x) + \epsilon)} \operatorname{id} d(\mu \circ x),$$

thus $\inf s(x) \leqslant E(x)(\mu) \leqslant \inf s(x) + \epsilon$, showing that $\inf s(x) = \inf\limits_{\mu \in \Omega_\sigma(L)} E(x)(\mu)$. Similarly, $\sup s(x) = \sup\limits_{\mu \in \Omega_\sigma(L)} E(x)(\mu)$. The final assertion follows from $\sup\limits_{\mu \in \Omega_\sigma(L)} |E(x)(\mu)| = \sup\limits_{\mu \in \operatorname{con}(\Omega_\sigma(L) \cup -\Omega_\sigma(L))} E(x)(\mu) = \|E(x)\|$.

Lemma 4.2: Let $x \in S(L)$ with $s(x) \subseteq [0, 1]$. Then

(i) $E(x)^{-1}(0) \cap \Omega_\sigma(L) = x\hat{\{0\}}$,

(ii) $E(x)^{-1}(1) \cap \Omega_\sigma(L) = x\hat{\{1\}}$.

Proof: (i): Let $E(x)(\mu) = 0, \mu \in \Omega_\sigma(L)$. Then $0 = \int \operatorname{id} d(\mu \circ x) = \int (\chi_{\{0\}} + \chi_{(0,1]} \operatorname{id} d(\mu \circ x) = \int \chi_{(0,1]} \operatorname{id} d(\mu \circ x)$. For $t \in (0, 1]$ $\chi_{(0,1]} \operatorname{id} > 0$, we conclude that $(\mu \circ x)(0, 1] = \mu(x(0, 1]) = 0$. Thus $\mu(x\{0\}) = 1$ since $(\mu \circ x)[0, 1] = 1$. Therefore $\mu \in x\hat{\{0\}}$. The converse is easily shown.

(ii): If $E(x)(\mu) = 1, \mu \in \Omega_\sigma(L)$ then $E(x)'(\mu) = \int (1 - t)\, d(\mu \circ x) = 0$. Now use similar arguments as in (i).

It follows from the foregoing observations that $E(x)$ ($x \in S^b(L)$) is a counter if and only if $s(x) \subseteq [0, 1]$. However, in general there are counters that do not arise in this manner. We get hold of a rather important class of orthomodular posets (with a strongly order determining $\Omega_\sigma(L)$) through the following definition.

The orthomodular poset $(L, \leqslant, ')$ is said to be *spectral* provided every counter is the expectation functional of some bounded observable. Without proof we note that the usual orthomodular poset of projections of a Hilbert space of dimension greater than 2 is spectral.

More insight is gained through

Theorem 4.3: *The following are mutually equivalent:*

(i) $(L, \leqslant, ')$ *is spectral,*

(ii) *to every* $f \in V_\sigma^*(L)$ *there exists* $x \in S^b(L)$ *such that* $f = E(x)$,

(iii) a) *if* $x, y \in S^b(L)$ *then there exists an observable* $z \in S^b(L)$ *such that* $E(z)(\mu) = E(x)(\mu) + E(y)(\mu)$ *for all* $\mu \in \Omega_\sigma(L)$ *and*

b) *if* $(x_\delta)_\delta$ *is a net of positive bounded observables such that* $\lim (E(x_\delta)(\mu))_\delta$ *exists for all* $\mu \in \Omega_\sigma(L)$ *and is uniformly bounded in* $\mu \in \Omega_\sigma(L)$ *then there exists an observable* $z \in S^b(L)$ *such that* $E(z)(\mu) = \lim (E(x_\delta)(\mu))_\delta$ *for* $\mu \in \Omega_\sigma(L)$.

Proof: (i) $\Rightarrow$ (ii): Let $g \in V_\sigma^*(L)$, we may assume $g \neq 0 = E(x_0)$. Then $1/2\,(g/\|g\| + e) \in [0, e]$. Now let $x \in S^b(L)$ be such that $E(x) = 1/2\,(g/\|g\| + e)$. The Borel function $\alpha(t) := 2\|g\|t - \|g\|$ is bounded on $s(x)$, hence $s(\alpha(x))$ is bounded showing that $\alpha(x) \in S^b(L)$. We now get for all $\mu \in \Omega_\sigma(L)$, using the "transformation theorem of integration" [9, p. 163], $E(\alpha(x))(\mu) = \int \mathrm{id}\, d(\mu \circ (x \circ \alpha^{-1})) = \int \mathrm{id}\, d((\mu \circ x) \circ \alpha^{-1}) = \int \alpha(t)\, d(\mu \circ x) = \int (2\|g\|t - \|g\|)\, d(\mu \circ x) = 2\|g\| \int \mathrm{id}\, d(\mu \circ x) - \|g\| = g(\mu)$. Therefore $E(\alpha(x)) = g$.

(ii) $\Rightarrow$ (i): This is immediate.

(ii) $\Rightarrow$ (iii): Clearly, condition a) holds since $V_\sigma^*(L)$ is a linear space. If $(E(x_\delta)(\mu))_\delta$ converges for all $\mu \in \Omega_\sigma(L)$ then it converges for all $\mu \in V_\sigma(L)$. Also the map $\mu \in V_\sigma(L) \to f(\mu) := \lim_\delta E(x_\delta)(\mu) \in \mathbb{R}$ is a linear functional on $V_\sigma(L)$. Since $0 \leqslant \inf s(x_\delta) \leqslant E(x_\delta)(\mu)$ and $f(\mu) \leqslant k \in \mathbb{R}$ for all $\mu \in \Omega_\sigma(L)$, we conclude that $f \in V_\sigma^*(L)$. Now $f = E(x)$ for some $x \in S^b(L)$ and assertion (b) follows.

(iii) $\Rightarrow$ (ii): First observe that $E := \{E(x) \mid x \in S^b(L)\}$ is a linear subspace of $V_\sigma^*(L)$ since $sE(x) = E(\alpha(x))$, $\alpha(t) = st\,(s \in \mathbb{R})$ and to $x, y \in S^b(L)$ there exists $z \in S^b(L)$ with $E(x) + E(y) = E(z)$. Also $P \subseteq E$, since $E(x_p) = f_p$ for all $p \in L$, and therefore pos $P \subseteq E$, pos P being the positive hull of P in $V_\sigma^*(L)$.

Let $(f_\delta)_\delta$ be a net in pos P that converges to $f \in V_\sigma^*(L)$ in the $\sigma(V_\sigma^*(L), V_\sigma(L))$-topology. To every f_δ there exists a positive observable $x_\delta \in S^b(L)$ (see lemma 4.1) with $f_\delta = E(x_\delta)$. Then clearly $\lim_\delta E(x_\delta)(\mu) = f(\mu) \leqslant \|f\|$ for all $\mu \in \Omega_\sigma(L)$ since $0 \leqslant f$. By condition b) there exists now an $x \in S^b(L)$ such that $E(x) = f$. Showing that $\sigma(V_\sigma^*(L), V_\sigma(L)) - \mathrm{cl}\,(\text{pos } P) \subseteq E$.

Consider now the dual pair $(V_\sigma^*(L), V_\sigma(L))$ with scalar product $(f, \mu) := f(\mu)$. By the bipolar theorem [11], we conclude that $(\text{pos } P)^{00} = \sigma(V_\sigma^*(L), V_\sigma(L)) - \text{cl}(\text{pos P})$, since pos P is convex and $0 \in \text{pos } P$. Since $s(\text{pos } P) = \text{pos } P$ for $s > 0$ we have $(\text{pos } P)^0 = \{\mu \in V_\sigma(L) \mid f(\mu) \leq 1 \text{ for all } f \in \text{pos } P\} = \{\mu \in V_\sigma(L) \mid f(\mu) \leq 0 \text{ for all } f \in \text{pos } P\}$. Then $(\text{pos } P)^0 = \{\mu \in V_\sigma(L) | \mu(p) \leq 0, \text{ all } p \in L\} = -K_\sigma(L)$ and finally $(\text{pos } P)^{00} = (-K_\sigma)^0 = \{f \in V_\sigma^*(L) \mid f \geq 0\}$. Combining with the above results, we get $\{f \in V_\sigma^*(L) \mid f \geq 0\} \subseteq E$. Since the set on the left-hand side is a generating cone we get $V_\sigma^*(L) \subseteq E$.

Spectrality implies that $(L, \leq, \perp)$ is isomorphic to the structure $(\mathcal{D}(L), \subseteq, \dagger)$:

Theorem 4.4: *Let $(L, \leq, ')$ be spectral. Then the map $p \in L \to \hat{p} \in \mathcal{D}(L)$ is surjective, $p \leq q \Leftrightarrow \hat{p} \subseteq \hat{q}$ and $p \perp q \Leftrightarrow \hat{p} \dagger \hat{q}$.*

Proof: Let $F \in \mathcal{D}(L)$ then there exists $f \in [0, e]$ such that $f^{-1}(1) \cap \Omega_\sigma(L) = F$. $(L, \leq, ')$ being spectral, there exists $x \in S^b(L)$ with $f = E(x)$. Since $s(x) \subseteq [0, 1]$ we conclude, by lemma 4.2, that $F = \hat{x}\{1\}$. The second assertion is immediate. Let $\hat{p} \dagger \hat{q}$. Then there is an $x \in S^b(L)$ with $E(x) \in [0, e]$, $\hat{p} \subseteq E(x)^{-1}(1) \cap \Omega_\sigma(L)$ and $\hat{q} \subseteq E(x)^{-1}(0) \cap \Omega_\sigma(L)$. Again by lemma 4.2, $\hat{p} \subseteq x\{1\}, \hat{q} \subseteq x\{0\}$ thus $p \leq x\{1\}$, $q \leq x\{0\}$. Therefore, $p \perp q$ since $\{1\} \cap \{0\} = \emptyset$.

Lemma 4.5: *If $(L, \leq, ')$ is spectral then $(L, \leq)$ is a lattice and for all $\mu \in \Omega_\sigma(L)$ $\mu(p) = \mu(q) = 1 \Rightarrow \mu(p \wedge q) = 1$.*

Proof: Let $p, q \in L$. By lemma 3.1 $(\mathcal{D}(L), \subseteq) - \inf\{\hat{p}, \hat{q}\}$ exists and equals $p \cap q$. Then clearly $p \wedge q$ exists, by the foregoing theorem, hence $(L, \subseteq)$ is a lattice. Also $(\hat{p} \wedge \hat{q}) = \hat{p} \cap \hat{q}$.

Theorem 4.3 shows that a spectral orthomodular poset satisfies the "existence condition" in the sense of S. P. Gudder, therefore the above result may be considered as an application of [7, lemma 1.4]. As observed in [8], combining these results with a theorem by the author [18, theorem 4.3] we obtain the

Theorem 4.6: *Let L be finite. Then $(L, \leq, ')$ is spectral if and only if $(L, \leq)$ is a Boolean lattice.*

We shall make use of the following technical

Lemma 4.7: Every monotonically decreasing net $(f_\delta)_\delta$ in $[0, e]$ has a $\sigma(V_\sigma^*(L), V_\sigma(L))$-limit f in $[0, e]$. Also $f \leq f_\delta$.

Proof: First note that $[0, e]$ is $\sigma(V_\sigma^*(L), V_\sigma(L))$-compact since it is the image of the unit ball in $V_\sigma^*(L)$ under the $\sigma(V_\sigma^*(L), V_\sigma(L))$-homeomorphism $g \to (g + e)/2$. Therefore there exists a subnet $(f_{\delta'})_{\delta'}$ and $f \in [0, e]$ such that $f = \sigma(V_\sigma^*(L), V_\sigma(L)) - \lim f_{\delta'}$. .

To this end, for every $\mu \in \Omega_\sigma(L)$, $(f_\delta(\mu))_\delta$ is a monotonically decreasing net in $[0, 1] \subseteq R$ and therefore converges. The subnet $(f_{\delta'}(\mu))_{\delta'}$ converges to $f(\mu)$, $\mu \in \Omega_\sigma(L)$. Hence $f_\delta(\mu) \to f(\mu)$ and $f(\mu) \leqslant f_\delta(\mu)$ for all $\mu \in \Omega_\sigma(L)$ and we are done. ·

Theorem 4.8: *Let $(L, \leqslant, ')$ be spectral. Then*

(i) *$(L, \leqslant)$ is a complete lattice,*

(ii) *for any family $(p_i)_{i \in I}$ in L and $\mu \in \Omega_\sigma(L)$: if $\mu(p_i) = 1$ for all $i \in I$ then $\mu(\bigwedge_{i \in I} p_i) = 1$.*

Proof: Let D be a non-empty subset of L and denote with $(D^f, \subseteq)$ the set of finite subsets of D directed by set-inclusion. By lemma 4.5, $\wedge C$ exists for $C \in D^f$. Then $(f_{\wedge C})_{C \in D^f}$ is a monotonically decreasing net in $[0, e]$ and, by the foregoing lemma, there exists $f \in [0, e]$ with $f \leqslant f_C$, $C \in D^f$, and $f = (V_\sigma^*(L), V_\sigma(L)) - \lim (f_{\wedge C})_{C \in D^f}$. Then in particular $f \leqslant f_p$ for all $p \in D$, thus $f^{-1}(1) \cap \Omega_\sigma(L) \subseteq \bigcap_{p \in D} \hat{p}$. Conversely, let $\mu \in \bigcap_{p \in D} \hat{p}$ then $f(\mu) = (\sigma(V_\sigma^*(L), V_\sigma(L) - \lim (f_{\wedge C})_{C \in D^f})(\mu) =$ $\lim (\mu(\wedge C))_{C \in D^f} = 1$, by the second statement in lemma 4.5. Consequently, $f^{-1}(1) \cap \Omega_\sigma(L) = \bigcap_{p \in D} \hat{p}$. Since $f^{-1}(1) \cap \Omega_\sigma(L) \in \mathcal{D}(L)$ there exists, by theorem 4.4 a proposition $q \in L$ with $\hat{q} = \bigcap_{p \in D} \hat{p}$. Then $\bigwedge_{p \in D} p$ exists and equals q again by theorem 4.4. The remainder follows immediately.

Corollary 4.9: *If $(L, \leqslant, ')$* is spectral then $(\mathcal{D}(L), \subseteq)$ a complete lattice and every semi-detectable property is a detectable property.

Theorem 4.10: *If $(L, \leqslant, ')$ is spectral then every σ-additive probability state is completely additive.*

Proof: The assertion holds true in any complete orthomodular lattice satisfying (ii) of theorem 4.8:

Let $E \in \mathcal{O}$ and $\mu \in \Omega_\sigma(L)$. Clearly, the net $(\mu(\vee C))_{C \in E^f}$ converges and $\mu(\vee C) = \sum_{p \in C} \mu(p)$ for $C \in E^f$. It is a well-known fact from real analysis that there exists a sequence $n \in N \to p_n \in E$ such that $\mu(p) = 0$ for all $p \in D := E - \{p_n \mid n \in N\}$ and $\lim (\sum_{p \in C} \mu(p))_{C \in E^f} = \sum_{n=1}^{\infty} \mu(p_n)$.

Now, since E is a maximal orthogonal set, we get $1 = \bigvee_{i=1}^{\infty} p_i \vee \bigvee D$ and therefore $\mu(1) = \mu(\bigvee_{n=1}^{\infty} p_n) + \mu(\bigvee D) = \sum_{n=1}^{\infty} \mu(p_n) + 0 = \lim(\mu(\bigvee C))_{C \in E f}$.

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