A SYMMETRY THEOREM ON A MODIFIED JEU DE TAQUIN

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Abstract. For their bijective proof of the hook-length formula for the number of standard tableaux of a fixed shape Novelli, Pak and Stoyanovskii [1] define a modified jeu de taquin for 'scrambled tableaux'. We extend their definition and present a nice symmetry theorem on this extension.

1. Introduction

A parition of a positive integer n is a sequence of integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ with $\lambda_1 + \lambda_2 + \cdots + \lambda_r = n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0$. The *(unshifted) Ferrers diagram* of shape λ is an array of cells with r left-justified rows and λ_i cells in row i. (See Figure 1.a.) If λ is a parition with distinct components then the *shifted Ferrers diagram* of shape λ is an array of cells with r rows, each row indented by one cell to the right with respect to the previous row and λ_i cells in row i. (See Figure 1.b.) If $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ are partitions (resp. partitions with distinct components) such that $\lambda_i \leq \mu_i$ for $1 \leq i \leq r$ then the unshifted (resp. shifted) skew Ferrer diagram of shape μ/λ is the diagram we obtain if we remove the cells of the unshifted (resp. shifted) Ferrers diagram of shape λ from the unshifted (resp. shifted) Ferrers diagram of shape μ. (See Figure 2.)

We clarify the notion of 'scrambled tableaux'.

Definition 1. *A tabloid of shape μ/λ is an arbitrary filling of the unshifted (resp. shifted) skew shape μ/λ with distinct integers $1, 2, \ldots, |\mu/\lambda|$.*

As usual a standard tableau is a tabloid with increasing rows and columns. For the rest of the article we fix an unshifted or shifted skew shape μ/λ and set $n = |\mu/\lambda|$.

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a. The Ferrers diagram corresponding to $(4, 3, 3, 1)$

b. The shifted Ferrers diagram corresponding to $(5, 4, 2, 1)$

Figure 1

a. The skew Ferrers diagram of shape $(5,5,4,3,2)/(4,4,1)$.

b. The shifted skew Ferrers diagram of shape $(7,6,4,2,1)/(5,3)$.

Figure 2

Example 1. *Observe that*

$$R = \quad \begin{array}{cccc} & & & 8 \\ & & 3 & 6 \\ 9 & 5 & 1 & 4 \\ & 2 & 7 & \end{array}$$

is a tabloid of shifted shape $(6,5,4,2)/(5,3)$ *and that*

$$P = \quad \begin{array}{cccc} & & & 2 \\ & & 1 & 5 \\ 3 & 4 & 6 & 8 \\ & 7 & 9 & \end{array}$$

is a standard tableau of shifted shape $(6,5,4,2)/(5,3)$.

Next we define forward jeu de taquin in a tabloid T of shape μ/λ. If (i,j) is a cell in the Ferrers diagram of shape μ/λ let $T_{i,j}$ denote its entry in T. In order to simplify the description of forward jeu de taquin we define $T_{i,j} = \infty$ if (i,j) is a cell outside of T.

Definition 2 (Forward jeu de taquin)**.** *Let T be a tabloid and e an entry in T. Forward jeu de taquin in T with e is defined as follows: Compare e with its neighbour to the right and with its neighbour below and if e is greater than the minimum of these neighbours we exchange e with the minimum. Next we compare e to its new neighbours to the right and below and exchange e with the minimum of these two if e is greater than this minimum. We repeat this procedure with e until e is smaller than its neighbour to the right and its neighbour below in T.*

Example 2. *Performing forward jeu de taquin with* 8 *in the tabloid*

$$\begin{array}{ccc} 8 & 1 & 4 \\ 2 & 3 & 5 \\ 6 & 7 & \end{array}$$

results in the sequence

$$\begin{array}{ccc} 1 & 8 & 4 \\ 2 & 3 & 5 \\ 6 & 7 & \end{array}, \qquad \begin{array}{ccc} 1 & 3 & 4 \\ 2 & 8 & 5 \\ 6 & 7 & \end{array}, \qquad \begin{array}{ccc} 1 & 3 & 4 \\ 2 & 5 & 8 \\ 6 & 7 & \end{array}.$$

Next we define an 'ordering procedure' which assigns to every tabloid T a standard tableau. The procedure depends on another standard tableau S.

Definition 3 (Modified jeu de taquin)**.** *Let T be a tabloid and S a standard tableau of the same shape. Modified jeu de taquin in T with respect to S is defined as step by step performing forward jeu de taquin with the entries in T in the order the standard tableau S predicts, starting with the entry in T, whose cell has the greatest label in S. We denote the resulting standard tableau by $\mathrm{MJ}_S(T)$.*

Observe that for tabloids of normal shape $(3,3,2)$ the modified jeu de taquin from [1] is the (extended) modified jeu de taquin with respect to the standard tableau $\begin{array}{ccc} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & \end{array}$ from Definition 3. There Novelli, Pak and Stoyanovskii show that their (restricted) modified jeu de taquin has the nice property that the number of tabloids that are mapped to a fixed standard tableau is independent of this output standard tableau. The aim of this paper is to present another property of the (extended) modified jeu de taquin.

In our running example (Example 1) modified jeu de taquin with respect to P in R gives the intermediate tabloids

$$\begin{array}{cccc} & & & 8 \\ & & 3 & 4 \\ 9 & 5 & 1 & 6 \\ & 2 & 7 & \end{array}, \qquad \begin{array}{cccc} & & & 8 \\ & & 3 & 4 \\ 9 & 1 & 5 & 6 \\ & 2 & 7 & \end{array}, \qquad \begin{array}{cccc} & & & 8 \\ & & 3 & 4 \\ 1 & 2 & 5 & 6 \\ & 7 & 9 & \end{array},$$

before we finally obtain

$$\mathrm{MJ}_P(R) = \begin{array}{cccc} & & & 4 \\ & & 3 & 6 \\ 1 & 2 & 5 & 8 \\ & 7 & 9 & \end{array} =: Q.$$

2. The symmerty theorem

If P, Q are two standard tableaux of the same shape μ/λ let $A_{P,Q}$ denote the number of tabloids T of shape μ/λ such that the application of modified jeu de taquin to T with respect to P results in Q. For standard tableaux of shape $(3,3,2)$ the matrix $(A_{P,Q})_{(P,Q)}$ is the following

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In this matrix 1 stands for 936, 2 stands for 944, 3 stands for 960, 4 stands for 976, 5 stands for 984 and 6 stands for 996. Moreover every row and column corresponds to one of the 42 standard tableaux of shape $(3,3,2)$, which are ordered lexicographically if we identify a standard tableau with the permutation we obtain by reading the standard tableau rowwise from top to bottom and within a row from left to right.

By this and other computer experiments we were led to the conjecture that $A_{P,Q} = A_{Q,P}$. However, we discovered that a more general theorem is the key for this observation. In order to state it, we need another definition.

Definition 4. *Let S be a standard tableau and $\pi \in \mathcal{S}_n$ a permutation. Then S_π denotes the tabloid we obtain from S by replacing the entry i in S by $\pi(i)$ for every i.*

Let $\pi = 3\,8\,9\,5\,6\,1\,2\,4\,7$. Observe that $R = P_\pi$ and

$$Q_{\pi^{-1}} = \quad \begin{array}{cccc} & & & 8 \\ & & 1 & 5 \\ 6 & 7 & 4 & 2 \\ & 9 & 3 & \end{array}.$$

The fact that $\mathrm{MJ}_Q(Q_{\pi^{-1}}) = P$ does not come by chance.

Theorem 1. *Let P, Q be two standard tableaux of the same shape and $\pi \in \mathcal{S}_n$ a permutation. Then*

$$\mathrm{MJ}_P(P_\pi) = Q \Leftrightarrow \mathrm{MJ}_Q(Q_{\pi^{-1}}) = P.$$

Before we are able to prove the theorem we need the definition of backward jeu de taquin which is in some sense the inverse operation of forward jeu de taquin. In order to simplify the description we set $T_{i,j} = 0$ if (i,j) is a cell outside of T, where T is a tabloid.

Definition 5 (Backward jeu de taquin)**.** *Let T be a tabloid and e an entry in T. Backward jeu de taquin in T with e is defined as follows: Exchange e with the maximum of its neighbour to the left and its neighbour above. We repeat this procedure with e until e has no neighbour to the left and no neighbour above in the fixed shape.*

Observe that in Example 2 the input tabloid can be obtained from the output tabloid by performing backward jeu de taquin with 8.

Proof of Theorem 1. We only have to show one direction of the assertion, for the other follows by symmetry.

Let S be a standard tableau and T be a tabloid. We define $\mathrm{FJ}(T,S) = (S', T')$, where $T' = \mathrm{MJ}_S(T)$ and S' is the tabloid we obtain from S after the process of performing modified jeu de taquin in T with respect to S, if we simultaneously apply the transpositions we apply during modified jeu de taquin to T also to S. (If we exchange $T_{i,j}$ and $T_{i,j+1}$ in T we exchange $S_{i,j}$ and $S_{i,j+1}$ in S and if we exchange $T_{i,j}$ and $T_{i+1,j}$ in T we exchange $S_{i,j}$ and $S_{i+1,j}$ in S.)

In our running example: If we perform modified jeu de taquin in R with respect to P and perform the transpositions simultaneously to P we obtain the intermediate tabloids

$$\begin{array}{cccc} & & & 2 \\ & & 1 & 8 \\ 3 & 4 & 6 & 5 \\ & 7 & 9 & \end{array}, \quad \begin{array}{cccc} & & & 2 \\ & & 1 & 8 \\ 3 & 6 & 4 & 5 \\ & 7 & 9 & \end{array}, \quad \begin{array}{cccc} & & & 2 \\ & & 1 & 8 \\ 6 & 7 & 4 & 5 \\ & 9 & 3 & \end{array}$$

before we finally obtain

$$\begin{array}{cccc} & & & 8 \\ & & 1 & 5 \\ 6 & 7 & 4 & 2 \\ & 9 & 3 & \end{array}.$$

Note that the output tabloid is equal to $Q_{\pi^{-1}}$. This is because in the course of applying FJ to a pair (P_π, P) the first tabloid of the current pair can always be obtained from the second tabloid in the pair by applying π.

Observe that the following operation BJ is the inverse of FJ. Let T' be a standard tableau and S' be a tabloid. For $i = 1$ to $i = n$ perform backward jeu de taquin in T' in the subshape of μ/λ consisting of the cells of S' whose entries are greater or equal than i with the entry of T' that is in the same cell than i in S' and perform the transpositions simultaneously in S' also. If T' results in T and S' results in S then we define $\mathrm{BJ}(S', T') = (T, S)$. Clearly S is a standard tableau by construction. Furthermore $\mathrm{BJ} \cdot \mathrm{FJ} = \mathrm{id}$ and $\mathrm{FJ} \cdot \mathrm{BJ} = \mathrm{id}$.

If we apply BJ to $T' = Q$ and $S' = Q_{\pi^{-1}}$ we obtain the following a pairs of intermediate tabloids:

$$\left(\begin{array}{cccc} & & & 2 \\ & & 1 & 8 \\ 6 & 7 & 4 & 5 \\ & 9 & 3 & \end{array}, \begin{array}{cccc} & & & 8 \\ & & 3 & 4 \\ 1 & 2 & 5 & 6 \\ & 7 & 9 & \end{array} \right), \left(\begin{array}{cccc} & & & 2 \\ & & 1 & 8 \\ 3 & 6 & 4 & 5 \\ & 7 & 9 & \end{array}, \begin{array}{cccc} & & & 8 \\ & & 3 & 4 \\ 9 & 1 & 5 & 6 \\ & 2 & 7 & \end{array} \right),$$

$$\left(\begin{array}{cccc} & & & 2 \\ & & 1 & 8 \\ 3 & 4 & 6 & 5 \\ & 7 & 9 & \end{array}, \begin{array}{cccc} & & & 8 \\ & & 3 & 4 \\ 9 & 5 & 1 & 6 \\ & 2 & 7 & \end{array} \right), \left(\begin{array}{cccc} & & & 2 \\ & & 1 & 5 \\ 3 & 4 & 6 & 8 \\ & 7 & 9 & \end{array}, \begin{array}{cccc} & & & 8 \\ & & 3 & 6 \\ 9 & 5 & 1 & 4 \\ & 2 & 7 & \end{array} \right)$$

$= (P, R)$

As in the example $\mathrm{MJ}_P(P_\pi) = Q$ implies $\mathrm{FJ}(P_\pi, P) = (Q_{\pi^{-1}}, Q)$. Thus the assertion of the theorem is that FJ is an involution. This is equivalent to $\mathrm{FJ} = \mathrm{BJ}$.

In order to show that we decompose FJ and BJ into its elementary steps. Let S, T be tabloids of the same shape and $1 \le i \le n$. Then the pair $(T', S') = J^i(T, S)$ is defined as follows: Let T' be the tabloid we obtain by performing forward jeu de taquin with the entry in the cell of T which is labelled with i in S and perform the corresponding transpositions in S also in order to obtain S'. Observe that

$$\mathrm{FJ}(T, S) = \mathrm{SWITCH}\, J^1 J^2 \ldots J^n(T, S)$$

with $\mathrm{SWITCH}(X, Y) = (Y, X)$ if T is a tabloid and S is a standard tableau. The pair $(S, T) = J_i(S', T')$ is defined as follows: Let T be the tabloid we obtain by performing backward jeu de taquin in T' in the subshape of μ/λ consisting of the cells of S' whose entries are greater or equal than i with the entry in the cell of T' which is labelled with i in S'. Again perform the corresponding transposition in S' also in order to obtain S. Observe that

$$\mathrm{BJ}(S',T') = \mathrm{SWITCH}\, J_n J_{n-1} \dots J_1(S',T')$$

if S' is a tabloid and T' is a standard tableau.

Next we show the following identity

$$\mathrm{FJ}(T,S) = \mathrm{SWITCH}\, J^1 \dots J^{k-1} J^{k+1} \dots J^n J_1(T,S), \tag{1}$$

for a tabloid T and a standard tableau S, where k is the entry of the cell in S which is labelled with 1 in T. Let (i,j) be the cell of 1 in T. If the cell (i,j) has no neighbour to the left and no neighbour above in T then $J_1(T,S) = (T,S)$ and since

$$J^k J^{k+1} \dots J^n(T,S) = J^{k+1} \dots J^n(T,S),$$

this proves the assertion in this case. Now suppose that $S_{i,j-1} > S_{i-1,j}$ and $(i,j-1),(i-1,j)$ are both cells in the fixed shape for the other cases are similar. Then 1 is first involved in a transposition in the application of FJ to (T,S) when performing the first step of $J^{S_{i,j-1}}$ to the current pair. In this case the entries in cells $(i,j-1)$ and (i,j) are exchanged in both tabloids. Note that at that point also the entry in cell $(i,j-1)$ is first involved in a transposition in FJ. But this tranposition is also the first step in the application of J_1 to (T,S). If $(i,j-1)$ has no neighbour to the left and no neighbour above in the fixed shape the assertion is proved, for 1 in T is neither involved in another transposition of FJ nor of J_1. Otherwise suppose $S_{i-1,j-1} > S_{i,j-2}$ and $(i-1,j-1),(i,j-2)$ are both cells in the fixed shape for the other cases are similar. Then 1 in T is involved in a transposition in the application of FJ to (T,S) for the second time when performing the first step of $J^{S_{i-1,j-1}}$ to the current pair. In this step the entries in cells $(i-1,j-1)$ and $(i,j-1)$ are exchanged in both tabloids. Again the entry in $(i-1,j-1)$ is involved in a transposition for the first time in FJ. But this transposition is also the second step in the application of J_1 to (T,S) etc. Roughly speaking the backward path in the application of J_1 to (T,S), which we obtain by performing backward jeu de taquin with the entry in S in the cell labelled with 1 in T, is equal to the 'backward path' of 1 in T, which we obtain indirectly in the application of FJ to (T,S) by performing forward jeu de taquin to all entries in T.

Now we show the assertion by induction with respect to n. For $n=1$ there is nothing to prove. Suppose that (T',S') is a pair of tabloids, where entry 1 in T' has no neighbour to the left and no neighbour above in T' and where S' without the entry in the cell of 1 in T' is standard. Let $\mathrm{FJ}'(T',S')$ and $\mathrm{BJ}'(T',S')$ denote the output pairs after the application of FJ and BJ to the pair (T',S') with the cell of 1 in T' omitted in both tabloids. By induction $\mathrm{FJ}'(T',S') = \mathrm{BJ}'(T',S')$. Thus

$$\mathrm{FJ}(T,S) = \mathrm{FJ}'(J_1(T,S)) = \mathrm{BJ}'(J_1(T,S)) = \mathrm{BJ}(T,S),$$

where the first equality follows from (1). □

Corollary 1. *Let P,Q be two standard tableaux of the same shape. Then $A_{P,Q} = A_{Q,P}$.*

Proof. By the theorem

$$\{\pi \in \mathcal{S}_n \,|\, \mathrm{MJ}_P(P_\pi) = Q\} = \{\pi \in \mathcal{S}_n \,|\, \mathrm{MJ}_Q(Q_{\pi^{-1}}) = P\}.$$

Thus

$$\begin{aligned} A_{P,Q} &= |\{T \text{ tabloid} \,|\, \mathrm{MJ}_P(T) = Q\}| \\ &= |\{\pi \in \mathcal{S}_n \,|\, \mathrm{MJ}_P(P_\pi) = Q\}| \\ &= |\{\pi \in \mathcal{S}_n \,|\, \mathrm{MJ}_Q(Q_{\pi^{-1}}) = P\}| \\ &= |\{T \text{ tabloid} \,|\, \mathrm{MJ}_Q(T) = P\}| \\ &= A_{Q,P}. \end{aligned}$$

□

References

[1] J.-C. Novelli, I. Pak and A.V. Stoyanovskii, A direct bijective proof of the hook-length formula, *Discrete Math. Theoret. Computer Science* **1** ()1997), 53–67.

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