A COMBINATORIAL PROOF OF THE HOOK-LENGTH FORMULA FOR SHIFTED STANDARD TABLEAUX

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Abstract. We present a combinatorial proof of the hook-length formula for shifted standard tableaux based on a modified jeu de taquin and the ideas of the bijective proof of the hook-length formula for ordinary standard tableaux by Novelli, Pak and Stoyanovskii [3].

1. Introduction

A partition of a positive integer n is a sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ with $\lambda_1 + \lambda_2 + \cdots + \lambda_r = n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$. The *(ordinary) Ferrers diagram* of shape λ is an array of cells with r left-justified rows and λ_i cells in row i. Figure 1.a shows the Ferrers diagram corresponding to $(4,3,3,1)$.

If λ is a partition with distinct components then the *shifted Ferrers diagram* of shape λ is an array of cells with r rows, each row indented by one cell to the right with respect to the previous row and λ_i cells in row i. Figure 1.b shows the shifted Ferrers diagram corresponding to $(5,4,2,1)$.

Given a partition λ of n, a *standard tableau*, respectively a *shifted standard tableau* of shape λ, is a filling of the cells of the ordinary Ferrers diagram, respectively shifted Ferrers diagram, of shape λ with $1, 2, \ldots, n$, such that the entries along rows and columns are increasing. Figure 2.a displays an example of a standard tableau of shape $(4,3,3,1)$ and Figure 2.b displays an example of a shifted standard tableau of shape $(5,4,2,1)$,

Once we have accepted these definitions it is a natural question to ask for the number of standard tableaux, respectively shifted standard tableaux, of a given shape λ. Surprisingly there exists a simple product formula for these numbers. It involves objects called *hooks*, which are defined in the following paragraph.

We label the cell in the i-th row and j-th column of the ordinary, respectively shifted, Ferrers diagram of shape λ by the pair (i,j). The *hook* of a cell (i,j) in an *ordinary* Ferrers diagram is the set of cells that are either in the same row as (i,j) and to the right of (i,j), or

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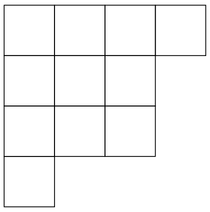
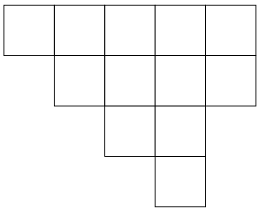


a. The Ferrers diagram corresponding to $(4,3,3,1)$



b. The shifted Ferrers diagram corresponding to $(5,4,2,1)$

Figure 1

a. A standard tableau

b. A shifted standard tableau

FIGURE 2

a. The hook of $(2,1)$

b. The shifted hook of $(1,2)$

FIGURE 3

in the same column as (i,j) and below (i,j), (i,j) included. The dots in Figure 3.a indicate the hook of the cell $(2,1)$. The *hook* of a cell (i,j) in a *shifted* Ferrers diagram again includes all cells that are either in the same row as (i,j) and to the right of (i,j), or in the same column as (i,j) and below (i,j), (i,j) included, but if this set contains the cell (j,j) on the main diagonal, then also the cells of the $(j+1)$-st row belong to the hook of (i,j). The dots in Figure 3.b indicate the hook of cell $(1,2)$. The *hook-length* $h_{i,j}$ of the cell (i,j) is the number of cells in the hook of (i,j).

Now we are in the position to state the hook-length formula.

Theorem 1. *The number of standard tableaux, respectively shifted standard tableaux, of shape λ is*

$$\frac{n!}{\prod_{(i,j)} h_{i,j}},$$

where the product in the denominator is taken over all cells in the Ferrers diagram, respectively shifted Ferrers diagram, of shape λ.

Thus the number of standard tableaux of shape $(4,3,3,1)$ is $\frac{11!}{7\cdot5\cdot4\cdot5\cdot3\cdot2\cdot4\cdot2} = 1188$ and the number of shifted standard tableaux of shape $(5,4,2,1)$ is $\frac{12!}{9\cdot7\cdot6\cdot5\cdot2\cdot6\cdot5\cdot4\cdot3\cdot2} = 176$.

For the history of the various proofs of the hook-length formula for ordinary standard tableaux see [5, page 400]. A recipe for an inductive proof of the hook-length formula for shifted standard tableaux can be found in [2, page 266]. In [4] it is shown that the nice probabilistic proof of the hook-length formula for ordinary standard tableaux in [1] has an analog for shifted standard tableaux.

A majority among the combinatorialists considers a bijective proof as the most aesthetic type of proof for an enumeration result. In [3] a bijective proof of the hook-length formula for ordinary standard tableaux based on a modified jeu de taquin is given. To my knowledge

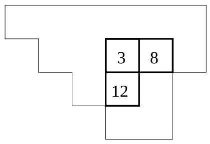


FIGURE 4. Entry $\rho = 3$ is stable, for $3 \leq 8, 12$.

there exists no bijective proof of the hook-length formula for shifted standard tableaux. The aim of this paper is to present a 'combinatorial' proof of the hook-length formula for shifted standard tableaux, which is in the spirit of the beautiful bijective proof in [3]. In the following paragraph we explain 'combinatorial' in this context.

If we discover two sets $\mathcal{S}, \mathcal{T}$ of combinatorial objects with the same cardinality, we believe that this fact is a projection of a canonical bijection between the two sets. Such a bijection is called a bijective proof of the equality $|\mathcal{S}| = |\mathcal{T}|$. Now suppose we are in the following more general situation: There exists an integer h such that $h \cdot |\mathcal{S}| = |\mathcal{T}|$. Then this fact could be a projection of a canonical h to 1 surjection from $\mathcal{T}$ onto $\mathcal{S}$, i.e. a map from $\mathcal{T}$ onto $\mathcal{S}$ where every element in $\mathcal{S}$ is assigned to exactly h elements in $\mathcal{T}$. In the following S_λ denotes the set of shifted standard tableaux of shape λ. A *shifted tabloid* of shape λ is an (arbitrary) filling of the cells of the shifted Ferrers diagram of shape λ with $1, 2, \ldots, n$. We denote the set of shifted tabloids of shape λ by T_λ and observe that its cardinality is $n!$. In our main theorem (Theorem 2) we present a $\prod_{(i,j)} h_{i,j}$ to 1 surjection from T_λ onto S_λ, which is clearly a proof of the hook-length formula for shifted standard tableaux.

2. Modified jeu de taquin

In this section we describe the sorting procedure which assigns to every 'scrambled' shifted tabloid $T \in T_\lambda$ a shifted standard tableau $S \in S_\lambda$. This map has the property that the number of shifted tabloids which are mapped to a fixed shifted standard tableau is $\prod_{(i,j)} h_{i,j}$. The sorting procedure is based on a modified jeu de taquin, which we have to describe first.

Notation. Let T be a shifted (hook) tabloid of shape λ. Then $T_{i,j} = T_{(i,j)}$ denotes the entry of cell (i,j). If the cell (i,j) does not exist in the shifted Ferrers diagram of shape λ, let $T_{i,j} = \infty$. For $1 \leq e \leq n$ there exists a cell (i,j) in the shifted Ferrers diagram of shape λ with $T_{i,j} = e$. We define $c_T(e) = (i,j)$.

Jeu de taquin in T with entry e and with respect to a set D. Let $1 \leq e \leq n$ and D be a set of cells in the shifted Ferrers diagram of shape λ. We define the routine *jeu de taquin in T with entry e and with respect to D* inductively. The output is another shifted tabloid U of shape λ. Let $c_T(e) = (i,j)$. Set $U = T$ and stop if either $(i,j) \in D$ or $T_{i,j} \leq \min(T_{i+1,j}, T_{i,j+1})$. (In the latter case e is said to be *stable*. See Figure 4.) Otherwise let $\rho \in \{(i+1,j), (i,j+1)\}$ be such that $T_\rho = \min(T_{i+1,j}, T_{i,j+1})$ and let T' denote the tabloid we obtain by exchanging the entries e and T_ρ in T. (See Figure 5.) Next perform jeu de taquin in T' with entry e and with respect to D in order to obtain the final tabloid U. (For an example see Figure 6.)

If D is the empty set we omit 'with respect to D'. The output tabloid U is denoted by $\mathrm{JT}_D(T,e)$ and the cell of e in U is denoted by $\mathrm{CJT}_D(T,e)$. If D is the k-th row of the shifted Ferrers diagram we write $\mathrm{JT}_k(T,e)$ and $\mathrm{CJT}_k(T,e)$, respectively.

The total order. We define a total order on the cells of a shifted Ferrers diagram of shape λ. A cell ρ_1 comes before cell ρ_2 if ρ_1 is in a lower row than ρ_2 or if both are in the

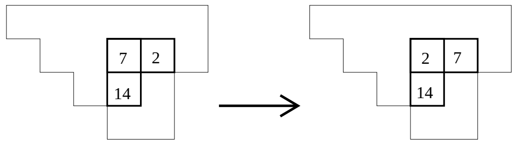


FIGURE 5. Entry $\rho = 7$ is unstable in the left tabloid and thus $2 = \min(14, 2)$ and 7 change place in the course of performing jeu de taquin with 7. Thus 2 is stable in the right tabloid. Next compare 7 to its new neighbours to the right and below...until 7 is either stable or the cell of 7 is in D.

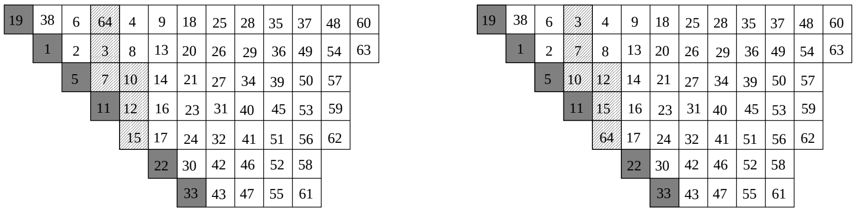


FIGURE 6. Performing jeu de taquin with $\rho = 64$ and with respect to the cells on the main diagonal in the tabloid on the left side results in the tabloid on the right side.

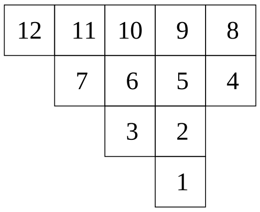


FIGURE 7. The total order.

same row but ρ_1 is to the right of ρ_2. Phrased differently, to obtain the total order one starts with the rightmost cell in the last row and reads each row from right to left, beginning with the bottom row and continuing up to the first row. Figure 7 displays this total order for the shifted Ferrers diagram of shape $(4, 3, 3, 1)$.

The $\prod_{(i,j)} h_{i,j}$ to 1 map from T_λ onto S_λ. Let T denote an arbitrary shifted tabloid of shape λ. In order to construct the corresponding shifted standard tableau S we perform step by step jeu de taquin with the entries of T subject to the total order we have just defined and starting with the entry in the smallest cell. To be more accurate:

$\rho =$ smallest cell with respect to the total order
$S = T$
Repeat
$S = \mathrm{JT}(S, S_\rho)$
$\rho =$ successor of ρ in the total order
until $\rho = (1, 1)$.

Example 1. *Consider the shifted tabloid*

$$T = \begin{array}{ccccc} 11 & 4 & 9 & 8 & 1 \\ & 12 & 6 & 2 & 3 \\ & & 10 & 5 & \\ & & & 7 & \end{array}$$

of shape $(5,4,2,1)$*. We construct the corresponding shifted standard tableau. According to the algorithm above we start with performing jeu de taquin with* 7*, but* 7 *is stable. The same is true for the entry* 5*. Performing jeu de taquin with* 10 *results in* $S = \begin{array}{ccccc} 11 & 4 & 9 & 8 & 1 \\ & 12 & 6 & 2 & 3 \\ & & 5 & 7 & \\ & & & 10 & \end{array}$*. Since* 3 *and* 2 *are stable the next change happens to be when performing jeu de taquin with entry* 6*.* $S = \begin{array}{ccccc} 11 & 4 & 9 & 8 & 1 \\ & 12 & 2 & 3 & 6 \\ & & 5 & 7 & \\ & & & 10 & \end{array}$*. We perform jeu de taquin with* 12 *—* $S = \begin{array}{ccccc} 11 & 4 & 9 & 8 & 1 \\ & 2 & 3 & 6 & 12 \\ & & 5 & 7 & \\ & & & 10 & \end{array}$*, — next with* 8 *—* $S = \begin{array}{ccccc} 11 & 4 & 9 & 1 & 8 \\ & 2 & 3 & 6 & 12 \\ & & 5 & 7 & \\ & & & 10 & \end{array}$*, — then with* 9 *—* $S = \begin{array}{ccccc} 11 & 4 & 1 & 6 & 8 \\ & 2 & 3 & 7 & 12 \\ & & 5 & 9 & \\ & & & 10 & \end{array}$*, — and with* 4 *—* $S = \begin{array}{ccccc} 11 & 1 & 3 & 6 & 8 \\ & 2 & 4 & 7 & 12 \\ & & 5 & 9 & \\ & & & 10 & \end{array}$*, — and finally with* 11 *—* $S = \begin{array}{ccccc} 1 & 2 & 3 & 6 & 8 \\ & 4 & 5 & 7 & 12 \\ & & 9 & 10 & \\ & & & 11 & \end{array}$*.*

Observe that the output tabloid S is a shifted standard tableau. We denote it by $\mathrm{STAND\_SPLIT}(T)$. We are in the position to state our main theorem. Note that the bijective proof in [3] shows that a similar theorem is true for ordinary standard tableaux.

Theorem 2. *The map* $T \to \mathrm{STAND\_SPLIT}(T)$ *is a* $\prod_{(i,j)} h_{i,j}$ *to* 1 *map from the set of shifted tabloids* T_λ *onto the set of shifted standard tableaux* S_λ*.*

As an interesting side-result we obtain a random algorithm which produces every shifted standard tableau of a given shape with the same probability.

Corollary 1. *The following algorithm produces every shifted standard tableaux of a given shape* λ *with the same probability.*

1. *Generate a permutation* π *of* $\{1,2,\ldots,n\}$ *subject to uniform distribution.*
2. *Construct the corresponding shifted tabloid* T_π *of shape* λ *by filling the elements from* π *into the shifted Ferrers diagram of shape* λ *rowwise from top to bottom and in each row from left to right.*
3. *Apply* $\mathrm{STAND\_SPLIT}(T_\pi)$ *in order to obtain the shifted standard tableau.*

We prove Theorem 2 by giving a rather complicated bijective proof of the hook-length formula. For that purpose we rewrite the hook-length formula as

$$n! = |S_\lambda| \cdot \prod_{(i,j)} h_{i,j}.$$

(2,2)	(1,2)	(1,5)	(4,4)	(2,5)
	(3,4)	(4,4)	(2,4)	(2,5)
		(3,4)	(4,4)	
			(4,4)	

FIGURE 8. A shifted hook tabloid of shape $(5,4,2,1)$.

We define combinatorial objects that correspond to $\prod_{(i,j)} h_{i,j}$ in this formula in our bijective proof. A *shifted hook tabloid* of shape λ is a filling of the cells of the shifted Ferrers diagram of shape λ with pairs of integers, such that the entry in a cell ρ are the coordinates of a cell in the hook of ρ. (See Figure 8. This definition of a shifted hook tabloid has a natural analog for ordinary Ferrers diagram, which is equivalent to the definition of a hook function in [3].) We denote the set of shifted hook tabloids by H_λ. Since $|H_\lambda| = \prod_{(i,j)} h_{i,j}$ it suffices to find a bijection between the set of shifted tabloids T_λ and the cartesian product of the set of shifted standard tableaux S_λ and the set of shifted hook tabloids H_λ to prove the hook-length formula for shifted standard tableaux.

$$T_\lambda \overset{\text{bijection}}{\leftrightarrow} S_\lambda \times H_\lambda$$

In order to prove Theorem 2 we construct such a bijection, where the shifted standard tableau is obtained from the shifted tabloid by the modified jeu de taquin described above. Our construction of the shifted hook tabloid for a given shifted tabloid is complicated and therefore this bijection probably does not meet the demands of a canonial bijection between T_λ and $S_\lambda \times H_\lambda$. However, it would be nice to find an easier-to-describe bijection. Also a shorter proof of Theorem 2 would be of interest. Maybe the present paper serves as an inspiration in this task. We should also mention that Novelli, Pak and Stoyanovskii claimed to have generalized their result to shifted standard tableaux in [3], but their result has not appeared yet.

We give some other definitions we need for the rest of the paper.

Reverse jeu de taquin in U with entry e and with respect to D. We define an inverse to jeu de taquin, which we need to construct the inverse of the function which assigns a pair of a shifted standard tableau and a shifted hook tabloid to a given shifted tabloid. In all algorithms where we use reverse jeu de taquin, we fix a row i. Reverse jeu de taquin depends on this row. The output of reverse jeu de taquin in U with entry e and with respect to D is again another shifted tabloid T of shape λ. Let $c_U(e) = (i', j')$. Set $T = U$ and stop if either $(i', j') = (i, i)$ or $(i', j') \in D$. Otherwise let $\rho \in \{(i'-1, j'), (i', j'-1)\}$ be such that $U_\rho = \max(U_{i'-1,j'}, U_{i',j'-1})$ if $i' \notin \{i, j'\}$, $\rho = (i', j'-1)$ if $i' = i$ and $\rho = (i'-1, j')$ if $i' = j'$ and let U' denote the tabloid we obtain by exchanging the entries e and U_ρ in U. Next perform jeu de taquin in U' with entry e and with respect to D in order to obtain the final tabloid T. (For an example read Figure 6 from right to left. The left shifted tabloid can be obtained from the right by performing reverse jeu de taquin with 64 and with respect to $(1,4)$.)

If D is the empty set we omit 'with respect to D'. In the following the output tabloid T is denoted by $\mathrm{RJT}_D(U, e)$ and the cell of e in T is denoted by $\mathrm{CRJT}_D(U, e)$. If D is the k-th row we write $\mathrm{RJT}_k(U, e)$ and $\mathrm{CRJT}_k(U, e)$, respectively.

Let T be a shifted tabloid of shape λ and $1 \le e \le n$. The *forward path of e in T with respect to D* is the set of cells e comes across when performing jeu de taquin in T with e and

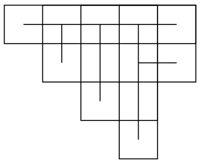


FIGURE 9. The backward paths of the entries in the shifted tabloid in Figure 2.b. Thus $1 <_T 3 <_T 2 <_T 6 <_T 5 <_T 4 <_T 12 <_T 11 <_T 9 <_T 10 <_T 7 <_T 8$.

with respect to D. For example the forward path of 64 with respect to the cells on the main diagonal in the left tabloid of Figure 6 is $\{(1,4),(2,4),(3,4),(3,5),(4,5),(5,5)\}$. Similarly the *backward path in T of e with respect to D* is the set of cells e comes across when performing reverse jeu de taquin in T with e and with respect to D. Clearly the backward path of 64 with respect to $(1,4)$ in the right shifted tabloid in Figure 6 coincides with the forward path in the left shifted tabloid.

We need one more definition before we are able to relate jeu de taquin and reverse jeu de taquin. Let T be a shifted tabloid of shape λ and ρ a cell in the shifted Ferrers diagram of shape λ. The shifted tabloid T is said to be *ordered up to cell ρ* if rows and columns are increasing in the subtabloid consisting of the cells that are smaller or equal to ρ with respect to the total order. (Note that in Figure 6 the left tabloid is ordered up to $(1,5)$.)

Observe that the routines *jeu de taquin* and *reverse jeu de taquin* are inverse to each other in the following sense: Let T be a shifted tabloid and e an entry in T with $c_T(e) = (i,j)$. Suppose T is ordered up to the predecessor of (i,j) in the total order. Perform jeu de taquin in T with e and with respect to D and obtain the tabloid U. If we perform reverse jeu de taquin in U with e and with respect to (i,j) (i being the fixed row), we reobtain T. In symbols

$$T = \mathrm{RJT}_{(i,j)}(\mathrm{JT}_D(T,e),e).$$

Furthermore observe that U is ordered up to (i,j) if $D = \emptyset$.

Conversely: Let e be an entry in row i or below of a shifted tabloid T that is ordered up to (i,j) and assume that the backward path of e in T contains (i,j) (i being the fixed row). Perform reverse jeu de taquin in T with e and respect to (i,j) and denote the output tabloid by U. If we perform jeu de taquin in U with e and with respect to $(i',j') = c_T(e)$, we reobtain T. In symbols

$$T = \mathrm{JT}_{(i',j')}(\mathrm{RJT}_{(i,j)}(T,e),e).$$

Furthermore observe that U is ordered up to the predecessor of (i,j). Therefore: *In order to be able to reconstruct T from* STAND_SPLIT(T) *we have to store the endcells of the forward paths we obtain in the application of modified jeu de taquin to T. The fact that these endcells are not independent from each other makes this storage non-trivial.*

Backward paths order. We define the *backward paths order* on the entries of a shifted tabloid T. Let e_1, e_2 be two entries in T and P_1, P_2 their backward paths (1 being the fixed row) in T. Furthermore let (i,j) denote the smallest cell in $P_1 \cap P_2$ with respect to the total order. If either $(i+1,j) \in P_1$ or $(i,j+1) \in P_2$ we define $e_1 <_T e_2$ and say that e_2 is greater than e_1 with respect to the backward paths order in T. For an example see Figure 9.

We introduce a manner-of-speaking: Let T be a shifted tabloid and H an accompanying partial shifted hook tabloid. Let ρ be an entry of a cell σ in the fixed i-th row of H, i.e.

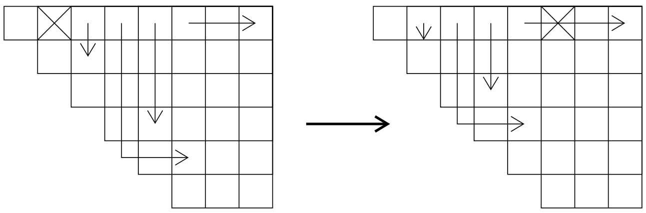


FIGURE 10. A shift from $(1,2)$ to $(1,6)$: The ends of the arrows indicate the entries of the cells at the origins of the arrows. The x denotes an empty cell.

$H_\sigma = \rho$. We say that T_ρ is a *horizontal candidate* in T with respect to H, if ρ is in the same column as σ in the shifted Ferrers diagram. For example consider the shifted hook tabloid in Figure 8 and an arbitrary shifted tabloid T of shape $(5,4,2,1)$. If $i=1$ then $T_{1,2}$, $T_{4,4}$ and $T_{2,5}$ are the horizontal candidates. If ρ is neither in the same row nor in the same column as σ, T_ρ is a *vertical candidate* with respect to H. In our example in Figure 8 $T_{2,2}$ is the only vertical candidate if $i=1$. Whenever in our algorithms we are looking for the cell (i,j) of a candidate e with $T_{H_{i,j}} = e$, we prefer the cell (if it exists) which is not in the same row or column as e.

The rest of the paper is organized as follows. In Section 3 we describe the Algorithm SPLIT that converts a shifted tabloid of shape λ into a pair of a shifted standard tableau of shape λ and a shifted hook tabloid of shape λ. In Section 4 we give some examples of the application of SPLIT. In Section 5 we describe the Algorithm MERGE that 'merges' a pair of a shifted standard tableau and a shifted hook tabloid to a shifted tabloid. In Section 6 we prove that the Algorithm MERGE is the inverse of the Algorithm SPLIT.

3. The Algorithm SPLIT

In this section we describe the algorithm SPLIT that transforms a shifted tabloid into a pair of a shifted standard tableau and a shifted hook tabloid.

The construction of the shifted hook tabloid is more involved compared to the construction of the shifted standard tabloid, which we have already described. It depends on the endcells of the paths we obtain in the course of performing jeu de taquin in the shifted tabloid and on the cells on the main diagonal these paths contain. We have to modify the order of some steps in the construction of the shifted standard tableau so that the shifted hook tabloid can be built simultaneously. But since these steps in question commute the modified algorithm for building the shifted standard tableau is equivalent to the origin algorithm.

Before we are in the position to describe the algorithm, we introduce two routines on a partial shifted hook tabloid H of shape λ.

A shift from cell (i,j) to cell (i,j') in a partial shifted hook tabloid. Let (i,j), (i,j') be two cells in H, $j \le j'$. We define the term *shift from (i,j) to (i,j') in H*. The output of this operation is another partial shifted hook tabloid H' which coincides with H except for the cells (i,k), $j \le k \le j'$. Let $j \le k < j'$. If $H_{i,k+1} = (i', k+1)$, $i < i'$, or $H_{i,k+1} = (k+2, j')$ then set $H'_{i,k} = (i'-1, k)$ or $H'_{i,k} = (k+1, j'-1)$, respectively. If $H_{i,k+1} = (i,l)$ then set $H'_{i,k} = (i,l)$. The cell (i,j') in H' remains empty. We denote H' by SHIFT(H,i,j,j'). For an example see Figure 10.

Whenever we perform jeu de taquin with an entry e and with respect to a set D in the shifted tabloid T in the algorithms below, we remember the end of the forward path in the

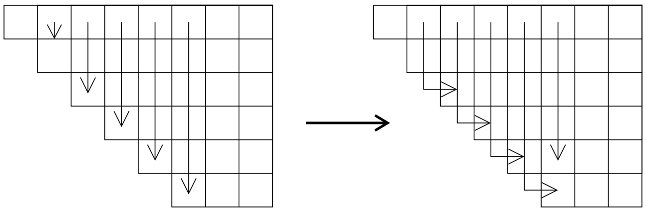


FIGURE 11. A transfer from $(1,2)$ to $(1,6)$: The ends of the arrows indicate the entries of the cells at the origins of the arrows.

accompanying shifted hook tabloid H. If we are in the course of performing jeu de taquin with the entries in the i-th row and (i',j') is the end of the forward path, this is either done by setting $H_{i,j'} = (i',j')$ or by setting $H_{i,i'-1} = (i',j')$ if the entry was a vertical candidate. The latter possibility is used if e was previously a vertical candidate. By reverse jeu de taquin the knowledge of the end of the forward path is enough to undo the performance of jeu de taquin. However, when $H_{i,j}$ or $H_{i,i'-1}$ is occupied we have to perform a shift from the cell in H that has previously pointed to e to either (i,j) or $(i,i'-1)$ in order to empty either (i,j) or $(i,i'-1)$. More accurate: If e is either a horizontal candidate or no candidate in (T,H) we denote by $\mathrm{JS}_D(T,H,e)$ the pair of a shifted tabloid T' and a shifted hook tabloid H', which is obtained in the following way: $T' = \mathrm{JT}_D(T,e)$, $H' = \mathrm{SHIFT}(H,i,q,j')$ and $H'_{i,j'} = (i',j')$, where q is the column of e in T and $c_{T'}(e) = (i',j')$. If e is a vertical candidate we denote by $\mathrm{JS}_D(T,H,e)$ the pair of a shifted tabloid T' and a shifted hook tabloid H', which is obtained in the following way: $T' = \mathrm{JT}_D(T,e)$, $H' = \mathrm{SHIFT}(H,i,p-1,i'-1)$ and $H'_{i,i'-1} = (i',j')$, where p is the row of e in T and $c_{T'}(e) = (i',j')$. If D is empty we write $\mathrm{JS}(T,H,e)$ and if D is the k-th row we write $\mathrm{JS}_k(T,H,e)$.

A transfer from cell (i,j) to cell (i,k) in a partial shifted hook tabloid. Let $j \le k \le r$ and $H_{i,j} = (i',j')$ with either $i' = i$ or $j' = j$. We define the term *transfer from cell* (i,j) *to cell* (i,k) in H. The output of this operation is again another partial shifted hook tabloid H' which coincides with H except for the cells (i,l), $j \le l \le k$. If $i' = i$ and $k \le j'$ let $H'_{i,k} = (i',j')$, otherwise let $H'_{i,k} = (i'+k-j',k)$. For $j \le l < k$ let $H'_{i,l} = (l+1,l+1)$. We denote H' by $\mathrm{TRANS}(H,i,j,k)$. For an example see Figure 11. Whenever we apply a transfer from (i,j) to (i,k) in H we have $H_{i,l} = (l,l)$ for $j < l \le k$ as in the example. This operation we need in the algorithms for changing horizontal candidates into vertical candidates.

The Algorithm SPLIT is divided into r steps, where in the i-th step we perform jeu de taquin with the entries in the $(r-i+1)$-st row. Within a row i, SPLIT is divided into 3 steps, SPLIT 1, SPLIT 2 and SPLIT 3. Assume we just start performing jeu de taquin with the entries in the i-th row. Let T denote the shifted tabloid we constructed so far and H the partial shifted hook tabloid. At this point T is ordered up to $(i+1,i+1)$ (as it is in the origin algorithm for constructing the shifted standard tableau in Section 2), the first i rows of H are empty and the last $r-i$ rows of H form a shifted hook tabloid. In SPLIT 1 we perform jeu de taquin with the entries in the i-th row and with respect to the set $MD = \{(1,1),(2,2),\dots,(r,r)\}$ of cells on the main diagonal.

SPLIT 1. Repeat for $j = \lambda_i + i - 1$ down to $j = i$: Set $(T,H) = \mathrm{JS}_{MD}(T,H,T_{i,j})$.

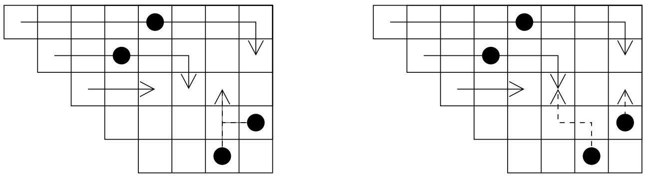


FIGURE 12. In the left tabloid we are in Case 1 of SPLIT 2 and in the right tabloid we are in Case 2 of SPLIT 2: In both diagrams $i=1$, $i'=h=3$ and the full lines indicate the forward paths in the course of constructing U. The circles indicates the horizontal candidates and if they are strictly below the 3-rd row the dashed lines indicate their backward paths with respect to the 3-rd row.

After SPLIT 1 for every entry e, whose forward path terminates in SPLIT 1 in a cell (k,k) on the main diagonal, we have $T_{k,k}=e$ and $H_{i,k}=(k,k)$ and therefore the unstable entries in T after the application of SPLIT 2 are horizontal candidates with respect to H. That is to say whenever a forward path of an entry $T_{i,j}$ ends in a cell (k,k) on the main diagonal, then the forwards paths of the following entries $T_{i,g}$, $g<j$ end strictly left of the k-th column. This is proved in Lemma 3.

SPLIT 2. Choose i', $i \le i' \le r$, maximal such that $H_{i,k}=(k,k)$ and $T_{k-1,k-1}$ is unstable (i.e. $T_{k-1,k-1} > T_{k-1,k}$) for $i < k \le i'$. If $i'=i$ jump to Case 2.

Set $U=T$. Repeat for $g=i'$ down to $g=i$: Set $U=\mathrm{JT}_{g+1}(U,T_{g,g})$.

We distinguish between two cases. Let h be minimal, $i \le h \le i'$, such that $T_{h,h}$ is not in the $(h+1)$-st row of U. We continue with Case 2 below if h does not exist. We also continue with Case 2 if $h=i'$ and there exists a horizontal candidate with respect to H strictly below the i'-th row of U which is smaller than $T_{i'-1,i'-1}$ with respect to the backward paths order in U. In all other cases we continue with Case 1. (See Figure 12.)

Reject the tabloid U we constructed so far in SPLIT 2.

Case 1. Repeat for $g=i'$ down to $g=h+1$: [Let k be such that either (g,k) or $(g+1,k)$ is the endcell of the forward path of $T_{g,g}$ in the procedure for constructing U. Set $T=\mathrm{JT}_{(g,k)}(T,T_{g,g})$ and $H_{i,g-1}=(g,k)$. (Note that the forward path of $T_{g,g}$ ends in (g,k).)]
Let $(h,k)=\mathrm{CJT}(T,T_{h,h})$ and set $T=\mathrm{JT}(T,T_{h,h})$. If $h-k \le i-i'$ let $H_{i,i'} = (i, i-h+k)$ otherwise let $H_{i,i'}=(i'+h-k,i')$.
Repeat for $g=h-1$ down to $g=i$: [Let $H_{i,g}=\mathrm{CJT}_{g+1}(T,T_{g,g})$ and $T=\mathrm{JT}_{g+1}(T,T_{g,g})$.]
Case 2. Set $(T,H)=\mathrm{JS}_{i'+1}(T,H,T_{i',i'})$.
Repeat for $g=i'-1$ down to $g=i$: [Let $H_{i,g}=\mathrm{CJT}_{g+1}(T,T_{g,g})$ and $T=\mathrm{JT}_{g+1}(T,T_{g,g})$.]

Observe the following (See also Figure 12.): Consider the backward paths with respect to the i'-th row of the horizontal candidates strictly below the i'-th row in the output tabloid. Then either one of these paths ends weakly to the left of the vertical candidate in the i'-th row or there exists no vertical candidate if and only if we were in Case 2 of SPLIT 2. (If we were in Case 1 of SPLIT 2 and the backward path of a horizontal candidate strictly below the i'-th row would end weakly to the left of the vertical candidate in the i'-th row after the application of Case 1 SPLIT 2 then either (if $h \neq i'$) the backward path of the same horizontal candidate includes the diagonal cell $(i'+1, i'+1)$ before the application of Case 1 in SPLIT 2 (by the argument in the proof of Lemma 2) which implies $H_{i,i'+1} = (i'+1, i'+1)$ (by property AFTER_SPLIT' 1 in the proof of Claim 3 in Section 6) and this is not possible by the choice of i' for $T_{i',i'}$ is unstable before the application of Case 1 in SPLIT 2 or $h = i'$ and therefore such a horizontal candidate does not exist in Case 1 of SPLIT 2 by the case distinction. Furthermore there always exists a vertical candidate after the performance of Case 1 of SPLIT 2.) In other words: We were in Case 2 of SPLIT 2 if and only if the smallest horizontal candidate strictly below the i'-th row is smaller than the smallest vertical candidate in the output pair.

For SPLIT 3 we need besides the pair (T, H) two sets of entries C' and C as input. The set C' contains the vertical candidates together with the smallest candidate with respect to the backward paths order if it is unstable. The set C contains the horizontal candidates on the main diagonal of T which are not in C'. Observe the entries in C' are strictly above of the entries in C, i.e. the row of the entry with maximal row in C' is smaller than the row of the entry with minimal row in C. This is also true in course of performing SPLIT 3. The same is true for the observation that all vertical candidates are strictly above of the entries in C.

SPLIT 3. Repeat the following until $C \cup C' = \emptyset$.
[If $C' = \emptyset$ choose $e \in C$ such that the row of e is minimal, set $C' = C' \cup \{e\}$ and $C = C \setminus \{e\}$. Choose $e \in C'$ with maximal row.
If $C \neq \emptyset$, choose $e' = T_{h,h} \in C$ such that h is minimal.
Consider the forward path P of e in T. We distinguish between three cases.
Case 1. P ends weakly left of $(h-1)$-st column or e' does not exist.
Set $(T, H) = \mathrm{JS}(T, H, e)$ and $C' = C' \setminus \{e\}$.
Case 2. $(h-1, h-1) \in P$ and $(h-1, h) \in P$.
Set $(T, H) = \mathrm{JS}_{(h-1,h-1)}(T, H, e)$.
Let $h \leq h'$ be maximal such that $T_{j-1,j-1}$ is unstable and $T_{j,j} \in C$ for $h < j \leq h'$.
If e is a horizontal candidate set $H = \mathrm{TRANS}(H, i, h-2, h'-1)$, otherwise $H = \mathrm{TRANS}(H, i, h-1, h')$.
Furthermore let $C' = C' \cup \{T_{h,h}, T_{h+1,h+1}, \ldots, T_{h',h'}\}$ and $C = C \setminus \{T_{h,h}, T_{h+1,h+1}, \ldots, T_{h',h'}\}$.
Case 3. P ends weakly right of column h, but does not contain $(h-1, h-1)$. Set $C' = C' \cup \{e'\}$ and $C = C \setminus \{e'\}$.]

In Figure 13 an example of Case 2 in SPLIT 3 is displayed.

Let T be a shifted tabloid ordered up to $(i+1, i+1)$ and H a partial shifted hook tabloid. Let (T', H') denote the pair we obtain after the application of SPLIT 1, SPLIT 2 and SPLIT 3 to the i-th row of (T, H). We denote $T' = \mathrm{STAND\_SPLIT}(T, H, i) = \mathrm{STAND\_SPLIT}(T, i)$ and $H' = \mathrm{HOOK\_SPLIT}(T, H, i)$. We are finally in the position to formulate the Algorithm SPLIT. The input is a shifted tabloid T of shape λ and an empty tabloid H of shape λ.

SPLIT. $T' = T$ and $H' = H$. Repeat for $i = r$ down to $i = 1$: [$H' = \mathrm{HOOK\_SPLIT}(T', H', i)$ and $T' = \mathrm{STAND\_SPLIT}(T', H', i)$.]

The output T' and H' of the algorithm is denoted by $\mathrm{STAND\_SPLIT}(T, H) = \mathrm{STAND\_SPLIT}(T)$ and $\mathrm{HOOK\_SPLIT}(T, H)$.

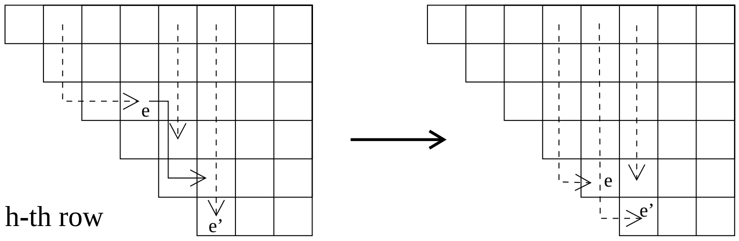


FIGURE 13. Situation before (left) and after (right) Case 2 in SPLIT 3. The dashed lines indicate relevant candidates with respect to the accompanying shifted hook tabloid (see also Figure 10). The full line indicates the forward path P of e.

4. EXAMPLES

In this section we give five examples for the application of SPLIT. All examples are of shape $(11,10,9,8,7,6,5,4,3,2)$ and the input shifted tabloid is ordered up to $(2,2)$. We perform jeu de taquin with the entries in the first row of the shifted tabloid and simultaneously built up the first row of the shifted hook tabloid. Our examples we cover all possible cases in SPLIT 2: In Example 2 we are in Case 2 of SPLIT 2 for h does not exist, in Example 3 we are in Case 2 of SPLIT 2 for $i=i'$, in Example 4 we are in Case 1 of SPLIT 2 for $h=1$, in Example 5 we are in Case 1 of SPLIT 2 with $h=i'$ and in Example 6 we are in Case 2 of SPLIT 2 with $h=i'$.

Example 2. *We consider the shifted tabloid*

$$T=\begin{array}{ccccccccccc}
63 & 64 & 61 & 41 & 34 & 20 & 62 & 65 & 56 & 15 & 54 \\
 & 1 & 2 & 3 & 4 & 7 & 8 & 13 & 16 & 18 & 24 \\
 & & 5 & 6 & 9 & 10 & 17 & 19 & 30 & 31 & 45 \\
 & & & 11 & 12 & 21 & 23 & 26 & 32 & 37 & 47 \\
 & & & & 14 & 22 & 25 & 27 & 36 & 39 & 51 \\
 & & & & & 28 & 29 & 35 & 40 & 44 & 52 \\
 & & & & & & 33 & 38 & 43 & 48 & 53 \\
 & & & & & & & 42 & 46 & 49 & 55 \\
 & & & & & & & & 50 & 57 & 58 \\
 & & & & & & & & & 59 & 60
\end{array}.$$

First we have to apply SPLIT 1 to the first row. If we perform jeu de taquin with 54 *and with respect to the cells on the main diagonal, the entry ends in cell* $(7,11)$.

$$T=\begin{array}{ccccccccccc}
63 & 64 & 61 & 41 & 34 & 20 & 62 & 65 & 56 & 15 & 24 \\
 & 1 & 2 & 3 & 4 & 7 & 8 & 13 & 16 & 18 & 45 \\
 & & 5 & 6 & 9 & 10 & 17 & 19 & 30 & 31 & 47 \\
 & & & 11 & 12 & 21 & 23 & 26 & 32 & 37 & 51 \\
 & & & & 14 & 22 & 25 & 27 & 36 & 39 & 52 \\
 & & & & & 28 & 29 & 35 & 40 & 44 & 53 \\
 & & & & & & 33 & 38 & 43 & 48 & 54 \\
 & & & & & & & 42 & 46 & 49 & 55 \\
 & & & & & & & & 50 & 57 & 58 \\
 & & & & & & & & & 59 & 60
\end{array}$$

Thus we set $H_{1,11} = (7, 11)$ and therefore the first row of the shifted hook tabloid is

$$H_1 = ((-,-),(-,-),(-,-),(-,-),(-,-),(-,-),(-,-),(-,-),(-,-),(-,-),(7,11))\ .$$

The entry 15 is stable, thus the shifted tabloid does not change in the next step and we set $H_{1,10} = (1, 10)$, the cell of 15.

$$H_1 = ((-,-),(-,-),(-,-),(-,-),(-,-),(-,-),(-,-),(-,-),(-,-),(1,10),(7,11))$$

Next we perform jeu de taquin with 56.

$$T = \begin{array}{ccccccccccc}
63 & 64 & 61 & 41 & 34 & 20 & 62 & 65 & 15 & 18 & 24 \\
 & 1 & 2 & 3 & 4 & 7 & 8 & 13 & 16 & 31 & 45 \\
 & & 5 & 6 & 9 & 10 & 17 & 19 & 30 & 37 & 47 \\
 & & & 11 & 12 & 21 & 23 & 26 & 32 & 39 & 51 \\
 & & & & 14 & 22 & 25 & 27 & 36 & 44 & 52 \\
 & & & & & 28 & 29 & 35 & 40 & 48 & 53 \\
 & & & & & & 33 & 38 & 43 & 49 & 54 \\
 & & & & & & & 42 & 46 & 55 & 56 \\
 & & & & & & & & 50 & 57 & 58 \\
 & & & & & & & & & 59 & 60
\end{array}$$

The forward path ends in cell $(8, 11)$ and therefore we want to set $H_{1,11} = (8, 11)$. However, $H_{1,11}$ is occupied, thus we perform a shift from $(1, 9)$ to $(1, 11)$.

$$H_1 = ((-,-),(-,-),(-,-),(-,-),(-,-),(-,-),(-,-),(-,-),(1,10),(6,10),(8,11))$$

If we perform jeu de taquin with 65 and with respect to the cells on the main diagonal, the entry gets stuck in $(9, 9)$ and remains unstable there.

$$T = \begin{array}{ccccccccccc}
63 & 64 & 61 & 41 & 34 & 20 & 62 & 13 & 15 & 18 & 24 \\
 & 1 & 2 & 3 & 4 & 7 & 8 & 16 & 30 & 31 & 45 \\
 & & 5 & 6 & 9 & 10 & 17 & 19 & 32 & 37 & 47 \\
 & & & 11 & 12 & 21 & 23 & 26 & 36 & 39 & 51 \\
 & & & & 14 & 22 & 25 & 27 & 40 & 44 & 52 \\
 & & & & & 28 & 29 & 35 & 43 & 48 & 53 \\
 & & & & & & 33 & 38 & 46 & 49 & 54 \\
 & & & & & & & 42 & 50 & 55 & 56 \\
 & & & & & & & & 65 & 57 & 58 \\
 & & & & & & & & & 59 & 60
\end{array}$$

Again we want to set $H_{1,9} = (9, 9)$, but $H_{1,9}$ is occupied and therefore we have to perform a shift.

$$H_1 = ((-,-),(-,-),(-,-),(-,-),(-,-),(-,-),(-,-),(1,10),(9,9),(6,10),(8,11))$$

Next we perform jeu de taquin with 62 and this entry again gets stuck in a cell on the main diagonal. Then we perform jeu de taquin with 20, 34, 41, 61, 64 and finally with 63. These entries either end in a stable position or on the main diagonal. Simultaneously we built up the first row of the shifted hook tabloid. There we put down the endcells of the paths in the appropriate columns and whenever we want to use a cell of the shifted hook tabloid which is *already occupied we perform the appropriate shift. This procedure results in*

$$T = \begin{array}{ccccccccccc}
63 & 1 & 2 & 3 & 4 & 7 & 8 & 13 & 15 & 18 & 24 \\
 & 64 & 5 & 6 & 9 & 10 & 16 & 19 & 30 & 31 & 45 \\
 & & 61 & 11 & 12 & 17 & 20 & 26 & 32 & 37 & 47 \\
 & & & 41 & 14 & 21 & 23 & 27 & 36 & 39 & 51 \\
 & & & & 34 & 22 & 25 & 35 & 40 & 44 & 52 \\
 & & & & & 28 & 29 & 38 & 43 & 48 & 53 \\
 & & & & & & 33 & 42 & 46 & 49 & 54 \\
 & & & & & & & 62 & 50 & 55 & 56 \\
 & & & & & & & & 65 & 57 & 58 \\
 & & & & & & & & & 59 & 60
\end{array}$$

and

$$H_1 = ((1,1),(2,2),(3,3),(4,4),(5,5),(1,10),(3,7),(8,8),(9,9),(6,10),(8,11)) \, .$$

We apply SPLIT 2 to this pair. Observe that $i' = 5$ *since* $H_{1,j} = (j,j)$ *for* $1 \le j \le 5$ *and* 63, 64, 61 *and* 41 *are unstable. We perform jeu de taquin with* 34, 41, 61, 64 *and* 63 *in that order and with repsect to the relative next row.*

$$U = \begin{array}{ccccccccccc}
1 & 2 & 3 & 4 & 7 & 8 & 13 & 15 & 18 & 24 & 45 \\
 & 5 & 6 & 9 & 10 & 16 & 19 & 30 & 31 & 37 & 63 \\
 & & 11 & 12 & 17 & 20 & 26 & 32 & 36 & 64 & 47 \\
 & & & 14 & 21 & 23 & 27 & 35 & 61 & 39 & 51 \\
 & & & & 22 & 25 & 29 & 41 & 40 & 44 & 52 \\
 & & & & & 28 & 34 & 38 & 43 & 48 & 53 \\
 & & & & & & 33 & 42 & 46 & 49 & 54 \\
 & & & & & & & 62 & 50 & 55 & 56 \\
 & & & & & & & & 65 & 57 & 58 \\
 & & & & & & & & & 59 & 60
\end{array}$$

We observe that every entry changes row and therefore h *does not exist. Consequently we are in Case 2 of SPLIT 2. We set* $T = U$*. Since* 34 *is in cell* $(6,7)$*, we set* $H_{1,7} = (6,7)$ *after performing a shift,* 41 *is in cell* $(5,8)$ *and therefore* $H_{1,4} = (5,8)$*,* 61 *is in cell* $(4,9)$ *and therefore* $H_{1,3} = (4,9)$*,* 64 *is in cell* $(3,10)$ *and therefore* $H_{1,2} = (3,10)$ *and* 63 *is in cell* $(2,11)$ *and therefore* $H_{1,1} = (2,11)$*. Thus*

$$H_1 = ((2,11),(3,10),(4,9),(5,8),(1,10),(2,6),(6,7),(8,8),(9,9),(6,10),(8,11)) \, .$$

Finally we apply SPLIT 3. Observe that $C' = \{34,41,61,64,63\}$ *and* $C = \{62,65\}$*. In the first step of SPLIT 3* $e = 34$ *and* $e' = 62$*. The forward path of* e *ends left of the column of* e' *and therefore we are in Case 1. We obtain*

$$T = \begin{array}{ccccccccccc}
1 & 2 & 3 & 4 & 7 & 8 & 13 & 15 & 18 & 24 & 45 \\
 & 5 & 6 & 9 & 10 & 16 & 19 & 30 & 31 & 37 & 63 \\
 & & 11 & 12 & 17 & 20 & 26 & 32 & 36 & 64 & 47 \\
 & & & 14 & 21 & 23 & 27 & 35 & 61 & 39 & 51 \\
 & & & & 22 & 25 & 29 & 41 & 40 & 44 & 52 \\
 & & & & & 28 & 33 & 38 & 43 & 48 & 53 \\
 & & & & & & 34 & 42 & 46 & 49 & 54 \\
 & & & & & & & 62 & 50 & 55 & 56 \\
 & & & & & & & & 65 & 57 & 58 \\
 & & & & & & & & & 59 & 60
\end{array}$$

and

$$H_1 = ((2,11),(3,10),(4,9),(5,8),(1,10),(2,6),(7,7),(8,8),(9,9),(6,10),(8,11))$$

for 34 *is stable in* $(7,7)$. *We delete* 34 *from* C': $C' = \{41, 61, 64, 63\}$ *and* $C = \{62, 65\}$.

In the next step of SPLIT 3 we have $e = 41$ *and* $e' = 62$. *Since the forward path of* 41 *ends weakly right of the column of* 62 *but does not contain* $(7,7)$, *we are in Case 3. Here we do not change* T *or* H, *we only delete* 62 *from* C *and put it into* C': $C' = \{62, 41, 61, 64, 63\}$ *and* $C = \{65\}$.

Now $e = 62$ *and* $e' = 65$ *and we are in Case 2 for the forward path of* 62 *contains* $(8,8)$ *and* $(8,9)$. *Therefore we want to change* 62 *into a vertical candidate by setting* $H_{1,7} = (8,8)$, *but since* $H_{1,7}$ *is occupied we perform a transfer from* $(1,7)$ *to* $(1,8)$.

$$H_1 = ((2,11),(3,10),(4,9),(5,8),(1,10),(2,6),(8,8),(8,8),(9,9),(6,10),(8,11))$$

The shifted tabloid T *does not change. Furthermore* $C' = \{65, 62, 41, 61, 64, 63\}$, $C = \emptyset$ *and we are always in Case 1 for the rest of the application of SPLIT 3.*

Next $e = 65$ *and its forward path ends in* $(10,11)$. *Therefore we want to set* $H_{1,11} = (10,11)$ *and since* $H_{1,11}$ *is occupied we perform a shift from the previous pointer* $(1,9)$ *of* 65 *to* $(1,11)$.

$$T = \begin{array}{ccccccccccc}
1 & 2 & 3 & 4 & 7 & 8 & 13 & 15 & 18 & 24 & 45 \\
 & 5 & 6 & 9 & 10 & 16 & 19 & 30 & 31 & 37 & 63 \\
 & & 11 & 12 & 17 & 20 & 26 & 32 & 36 & 64 & 47 \\
 & & & 14 & 21 & 23 & 27 & 35 & 61 & 39 & 51 \\
 & & & & 22 & 25 & 29 & 41 & 40 & 44 & 52 \\
 & & & & & 28 & 33 & 38 & 43 & 48 & 53 \\
 & & & & & & 34 & 42 & 46 & 49 & 54 \\
 & & & & & & & 62 & 50 & 55 & 56 \\
 & & & & & & & & 57 & 58 & 60 \\
 & & & & & & & & & 59 & 65
\end{array}$$

We have

$$H_1 = ((2,11),(3,10),(4,9),(5,8),(1,10),(2,6),(8,8),(8,8),(5,9),(7,10),(10,11))$$

and $C' = \{62, 41, 61, 64, 63\}$.

The forward path of $e = 62$ *ends in* $(9,11)$. *Therefore we want to set* $H_{1,8} = (9,11)$ *and since* $H_{1,8}$ *is occupied we perform a shift from* $(1,7)$ *to* $(1,8)$. *This yields*

$$T = \begin{array}{ccccccccccc}
1 & 2 & 3 & 4 & 7 & 8 & 13 & 15 & 18 & 24 & 45 \\
 & 5 & 6 & 9 & 10 & 16 & 19 & 30 & 31 & 37 & 63 \\
 & & 11 & 12 & 17 & 20 & 26 & 32 & 36 & 64 & 47 \\
 & & & 14 & 21 & 23 & 27 & 35 & 61 & 39 & 51 \\
 & & & & 22 & 25 & 29 & 41 & 40 & 44 & 52 \\
 & & & & & 28 & 33 & 38 & 43 & 48 & 53 \\
 & & & & & & 34 & 42 & 46 & 49 & 54 \\
 & & & & & & & 50 & 55 & 56 & 60 \\
 & & & & & & & & 57 & 58 & 62 \\
 & & & & & & & & & 59 & 65
\end{array}.$$

Moreover

$$H_1 = ((2,11),(3,10),(4,9),(5,8),(1,10),(2,6),(7,7),(9,11),(5,9),(7,10),(10,11))$$

and $C' = \{41, 61, 64, 63\}$.

In the next step $e = 41$ *etc.* ...

Finally we obtain the following shifted standard tableau

$$T = \begin{array}{ccccccccccc}
1 & 2 & 3 & 4 & 7 & 8 & 13 & 15 & 18 & 24 & 45 \\
 & 5 & 6 & 9 & 10 & 16 & 19 & 30 & 31 & 37 & 47 \\
 & & 11 & 12 & 17 & 20 & 26 & 32 & 36 & 44 & 51 \\
 & & & 14 & 21 & 23 & 27 & 35 & 39 & 48 & 52 \\
 & & & & 22 & 25 & 29 & 38 & 40 & 49 & 60 \\
 & & & & & 28 & 33 & 41 & 43 & 53 & 61 \\
 & & & & & & 34 & 42 & 46 & 54 & 62 \\
 & & & & & & & 50 & 55 & 56 & 63 \\
 & & & & & & & & 57 & 58 & 64 \\
 & & & & & & & & & 59 & 65
\end{array}$$

and the following first row of H

$$H_1 = ((1,10),(3,5),(1,5),(4,4),(6,9),(7,9),(8,11),(9,11),(5,9),(7,10),(10,11)) \,.$$

Example 3. *We consider*

$$T = \begin{array}{ccccccccccc}
64 & 35 & 8 & 12 & 36 & 49 & 51 & 1 & 34 & 63 & 54 \\
 & 2 & 3 & 5 & 6 & 10 & 13 & 14 & 15 & 20 & 27 \\
 & & 4 & 7 & 9 & 11 & 16 & 21 & 25 & 30 & 31 \\
 & & & 17 & 18 & 19 & 24 & 28 & 29 & 37 & 41 \\
 & & & & 22 & 23 & 26 & 33 & 42 & 43 & 45 \\
 & & & & & 32 & 38 & 40 & 46 & 47 & 53 \\
 & & & & & & 39 & 44 & 50 & 52 & 58 \\
 & & & & & & & 48 & 55 & 56 & 61 \\
 & & & & & & & & 57 & 59 & 62 \\
 & & & & & & & & & 60 & 65
\end{array} \,.$$

After the application of SPLIT 1 we obtain the pair

$$T = \begin{array}{ccccccccccc}
64 & 1 & 3 & 5 & 6 & 10 & 13 & 14 & 15 & 20 & 27 \\
 & 2 & 4 & 7 & 9 & 11 & 16 & 21 & 25 & 30 & 31 \\
 & & 35 & 8 & 12 & 19 & 24 & 28 & 29 & 37 & 41 \\
 & & & 17 & 18 & 23 & 26 & 33 & 34 & 43 & 45 \\
 & & & & 22 & 32 & 38 & 40 & 42 & 47 & 53 \\
 & & & & & 36 & 39 & 44 & 46 & 52 & 54 \\
 & & & & & & 49 & 48 & 50 & 56 & 58 \\
 & & & & & & & 51 & 55 & 59 & 61 \\
 & & & & & & & & 57 & 60 & 62 \\
 & & & & & & & & & 63 & 65
\end{array} \,,$$

$$H_1 = ((1,1),(1,8),(3,3),(3,4),(3,5),(6,6),(7,7),(8,8),(4,9),(10,10),(6,11)) \,.$$

Since $i' = i$ we are in Case 2 of SPLIT 2. We move 64 *to the second row*

$$T = \begin{array}{ccccccccccc}
1 & 2 & 3 & 5 & 6 & 10 & 13 & 14 & 15 & 20 & 27 \\
 & 64 & 4 & 7 & 9 & 11 & 16 & 21 & 25 & 30 & 31 \\
 & & 35 & 8 & 12 & 19 & 24 & 28 & 29 & 37 & 41 \\
 & & & 17 & 18 & 23 & 26 & 33 & 34 & 43 & 45 \\
 & & & & 22 & 32 & 38 & 40 & 42 & 47 & 53 \\
 & & & & & 36 & 39 & 44 & 46 & 52 & 54 \\
 & & & & & & 49 & 48 & 50 & 56 & 58 \\
 & & & & & & & 51 & 55 & 59 & 61 \\
 & & & & & & & & 57 & 60 & 62 \\
 & & & & & & & & & 63 & 65
\end{array}$$

and set $H_{1,2} = (2,2)$ after performing the appropriate shift

$$H_1 = ((1,8),(2,2),(3,3),(3,4),(3,5),(6,6),(7,7),(8,8),(4,9),(10,10),(6,11))\,.$$

We set $C' = \{64\}$ and $C = \{35,36,49,51,63\}$. In the first step of SPLIT 3 we have $e = 64$ and $e' = 35$. Since the forward path of 64 *includes $(2,2)$ and $(2,3)$ we are in Case 2 of SPLIT 3. The shifted tabloid T does not change, but we want to set $H_{1,1} = (2,2)$ and therefore perform a transfer from $(1,1)$ to $(1,2)$ in H.*

$$H_1 = ((2,2),(1,8),(3,3),(3,4),(3,5),(6,6),(7,7),(8,8),(4,9),(10,10),(6,11))$$

Furthermore $C' = \{35,64\}$ and $C = \{36,49,51,63\}$.

Next $e = 35$ and $e' = 36$. Again we are in Case 2.

$$T = \begin{array}{ccccccccccc}
1 & 2 & 3 & 5 & 6 & 10 & 13 & 14 & 15 & 20 & 27 \\
 & 64 & 4 & 7 & 9 & 11 & 16 & 21 & 25 & 30 & 31 \\
 & & 8 & 12 & 18 & 19 & 24 & 28 & 29 & 37 & 41 \\
 & & & 17 & 22 & 23 & 26 & 33 & 34 & 43 & 45 \\
 & & & & 35 & 32 & 38 & 40 & 42 & 47 & 53 \\
 & & & & & 36 & 39 & 44 & 46 & 52 & 54 \\
 & & & & & & 49 & 48 & 50 & 56 & 58 \\
 & & & & & & & 51 & 55 & 59 & 61 \\
 & & & & & & & & 57 & 60 & 62 \\
 & & & & & & & & & 63 & 65
\end{array}$$

We set $H_{1,4} = (5,5)$ after performing an appropriate shift and a transfer from $(1,4)$ to $(1,5)$.

$$H_1 = ((2,2),(1,8),(2,3),(5,5),(3,5),(6,6),(7,7),(8,8),(4,9),(10,10),(6,11))$$

We update $C' = \{36,35,64\}$ and $C = \{49,51,63\}$.

Next $e = 36$ and $e' = 49$ and since 36 *is stable we are in Case 1. Thus T and H do not change, but $C' = \{35,64\}$ and $C = \{49,51,63\}$.*

Therefore $e = 35$ *and* $e' = 49$ *and we are in Case 1.*

$$T = \begin{array}{ccccccccccc}
1 & 2 & 3 & 5 & 6 & 10 & 13 & 14 & 15 & 20 & 27 \\
 & 64 & 4 & 7 & 9 & 11 & 16 & 21 & 25 & 30 & 31 \\
 & & 8 & 12 & 18 & 19 & 24 & 28 & 29 & 37 & 41 \\
 & & & 17 & 22 & 23 & 26 & 33 & 34 & 43 & 45 \\
 & & & & 32 & 35 & 38 & 40 & 42 & 47 & 53 \\
 & & & & & 36 & 39 & 44 & 46 & 52 & 54 \\
 & & & & & & 49 & 48 & 50 & 56 & 58 \\
 & & & & & & & 51 & 55 & 59 & 61 \\
 & & & & & & & & 57 & 60 & 62 \\
 & & & & & & & & & 63 & 65
\end{array}$$

$$H_1 = ((2,2),(1,8),(2,3),(5,6),(3,5),(6,6),(7,7),(8,8),(4,9),(10,10),(6,11))$$

Moreover $C' = \{64\}$ *and* $C = \{49, 51, 63\}$.

In the next step $e = 64$ *and* $e' = 49$. *Because of the run of the forward path of* 64 *we are in Case 3. Thus* T *and* H *do not change, only* $C' = \{49, 64\}$ *and* $C = \{51, 63\}$.

Therefore $e = 49$ *and* $e' = 51$. *We are in Case 2 and* T *does not change. We want to set* $H_{1,6} = (7,7)$. *Since* $H_{1,6}$ *is occupied by a pointer which does not point to* 49, *we perform a transfer from* $(1,6)$ *to* $(1,7)$.

$$H_1 = ((2,2),(1,8),(2,3),(5,6),(3,5),(7,7),(7,7),(8,8),(4,9),(10,10),(6,11))$$

Moreover $C' = \{51, 49, 64\}$ *and* $C = \{63\}$.

Next $e = 51$ *and* $e' = 63$. *We are in Case 1 for* 51 *is stable and thus* $C' = \{49, 64\}$ *and* $C = \{63\}$.

Therefore $e = 49$ *and* $e' = 63$. *Again we are in Case 1.*

$$T = \begin{array}{ccccccccccc}
1 & 2 & 3 & 5 & 6 & 10 & 13 & 14 & 15 & 20 & 27 \\
 & 64 & 4 & 7 & 9 & 11 & 16 & 21 & 25 & 30 & 31 \\
 & & 8 & 12 & 18 & 19 & 24 & 28 & 29 & 37 & 41 \\
 & & & 17 & 22 & 23 & 26 & 33 & 34 & 43 & 45 \\
 & & & & 32 & 35 & 38 & 40 & 42 & 47 & 53 \\
 & & & & & 36 & 39 & 44 & 46 & 52 & 54 \\
 & & & & & & 48 & 49 & 50 & 56 & 58 \\
 & & & & & & & 51 & 55 & 59 & 61 \\
 & & & & & & & & 57 & 60 & 62 \\
 & & & & & & & & & 63 & 65
\end{array}$$

$$H_1 = ((2,2),(1,8),(2,3),(5,6),(3,5),(7,8),(7,7),(8,8),(4,9),(10,10),(6,11))$$

Moreover $C' = \{64\}$ *and* $C = \{63\}$.

Now we have $e = 64$ *and* $e = 63$. *We are in Case 2.*

$$T = \begin{array}{ccccccccccc}
1 & 2 & 3 & 5 & 6 & 10 & 13 & 14 & 15 & 20 & 27 \\
 & 4 & 7 & 9 & 11 & 16 & 21 & 25 & 29 & 30 & 31 \\
 & & 8 & 12 & 18 & 19 & 24 & 28 & 34 & 37 & 41 \\
 & & & 17 & 22 & 23 & 26 & 33 & 42 & 43 & 45 \\
 & & & & 32 & 35 & 38 & 40 & 46 & 47 & 53 \\
 & & & & & 36 & 39 & 44 & 50 & 52 & 54 \\
 & & & & & & 48 & 49 & 55 & 56 & 58 \\
 & & & & & & & 51 & 57 & 59 & 61 \\
 & & & & & & & & 64 & 60 & 62 \\
 & & & & & & & & & 63 & 65
\end{array}$$

Since 64 *was already a vertical candidate we want to set* $H_{1,9}=(10,10)$*. For that purpose we perform a transfer from* $(1,9)$ *to* $(1,10)$*.*

$$H_1=((1,8),(1,2),(4,5),(2,4),(6,7),(6,8),(7,7),(9,9),(10,10),(5,10),(6,11))$$

We update $C'=\{63,64\}$ *and* $C=\emptyset$*. Now* 63 *is stable and* $C'=\{64\}$ *after the next step. We terminate with the pair*

$$T=\begin{array}{ccccccccccc}
1 & 2 & 3 & 5 & 6 & 10 & 13 & 14 & 15 & 20 & 27\\
 & 4 & 7 & 9 & 11 & 16 & 21 & 25 & 29 & 30 & 31\\
 & & 8 & 12 & 18 & 19 & 24 & 28 & 34 & 37 & 41\\
 & & & 17 & 22 & 23 & 26 & 33 & 42 & 43 & 45\\
 & & & & 32 & 35 & 38 & 40 & 46 & 47 & 53\\
 & & & & & 36 & 39 & 44 & 50 & 52 & 54\\
 & & & & & & 48 & 49 & 55 & 56 & 58\\
 & & & & & & & 51 & 57 & 59 & 61\\
 & & & & & & & & 60 & 62 & 64\\
 & & & & & & & & & 63 & 65
\end{array},$$

$$H_1=((1,8),(1,2),(4,5),(2,4),(6,7),(6,8),(7,7),(9,11),(10,10),(5,10),(6,11))\ .$$

Example 4. *Let*

$$T=\begin{array}{ccccccccccc}
22 & 15 & 37 & 21 & 44 & 11 & 62 & 14 & 57 & 13 & 34\\
 & 1 & 2 & 4 & 6 & 7 & 10 & 17 & 19 & 28 & 32\\
 & & 3 & 9 & 8 & 9 & 23 & 27 & 31 & 38 & 41\\
 & & & 12 & 16 & 18 & 24 & 29 & 36 & 40 & 53\\
 & & & & 20 & 25 & 26 & 30 & 46 & 47 & 54\\
 & & & & & 33 & 35 & 42 & 48 & 50 & 55\\
 & & & & & & 39 & 43 & 49 & 51 & 60\\
 & & & & & & & 45 & 52 & 58 & 63\\
 & & & & & & & & 56 & 59 & 64\\
 & & & & & & & & & 61 & 65
\end{array}.$$

After the application of SPLIT 1 we obtain

$$T=\begin{array}{ccccccccccc}
22 & 1 & 2 & 4 & 6 & 7 & 10 & 13 & 14 & 28 & 32\\
 & 15 & 3 & 9 & 8 & 9 & 17 & 19 & 31 & 34 & 41\\
 & & 37 & 11 & 16 & 18 & 23 & 27 & 36 & 38 & 53\\
 & & & 12 & 20 & 24 & 26 & 29 & 40 & 47 & 54\\
 & & & & 21 & 25 & 30 & 42 & 46 & 50 & 55\\
 & & & & & 33 & 35 & 43 & 48 & 51 & 57\\
 & & & & & & 39 & 44 & 49 & 58 & 60\\
 & & & & & & & 45 & 52 & 59 & 63\\
 & & & & & & & & 56 & 61 & 64\\
 & & & & & & & & & 62 & 65
\end{array}$$

and the first row of the shifted hook tabloid is

$$H_1=((1,1),(2,2),(3,3),(1,4),(5,5),(1,10),(1,9),(7,8),(1,10),(10,10),(6,11))\ .$$

We apply SPLIT 2 to the pair. Observe that $i'=3$ *since* $H_{1,j}=(j,j)$ *for* $1\le j\le 3$ *and the entries* 22*,* 15 *are unstable.*

If we perform jeu de taquin with 37*,* 15 *and* 22 *and with respect to the relative next row we observe that* 22 *does not change row and thus* $h=1$*. Therefore we are in Case 1 of SPLIT 2.*

We perform jeu de taquin with 37, 15 *and* 22 *in that order and with respect to the last cell in the forward path in the course of constructing* U *of the current row. We obtain*

$$T = \begin{array}{ccccccccccc}
1 & 2 & 4 & 6 & 7 & 10 & 13 & 14 & 22 & 28 & 32 \\
 & 3 & 9 & 8 & 9 & 15 & 17 & 19 & 31 & 34 & 41 \\
 & & 11 & 37 & 16 & 18 & 23 & 27 & 36 & 38 & 53 \\
 & & & 12 & 20 & 24 & 26 & 29 & 40 & 47 & 54 \\
 & & & & 21 & 25 & 30 & 42 & 46 & 50 & 55 \\
 & & & & & 33 & 35 & 43 & 48 & 51 & 57 \\
 & & & & & & 39 & 44 & 49 & 58 & 60 \\
 & & & & & & & 45 & 52 & 59 & 63 \\
 & & & & & & & & 56 & 61 & 64 \\
 & & & & & & & & & 62 & 65
\end{array} .$$

Since 37 *moved to cell* $(3,4)$ *we set* $H_{1,2} = (3,4)$*, since* 15 *moved to* $(2,6)$ *we set* $H_{1,1} = (2,6)$ *and since* 22 *moved to* $(1,9)$ *we set* $H_{1,3} = (1,9)$*.*

$$H_1 = ((2,6),(3,4),(1,9),(1,4),(5,5),(1,10),(1,9),(7,8),(1,10),(10,10),(6,11))$$

After the application of SPLIT 3 we obtain

$$T = \begin{array}{ccccccccccc}
1 & 2 & 4 & 6 & 7 & 10 & 13 & 14 & 22 & 28 & 32 \\
 & 3 & 9 & 8 & 9 & 15 & 17 & 19 & 31 & 34 & 41 \\
 & & 11 & 12 & 16 & 18 & 23 & 27 & 36 & 38 & 53 \\
 & & & 20 & 21 & 24 & 26 & 29 & 40 & 47 & 54 \\
 & & & & 25 & 30 & 35 & 42 & 46 & 50 & 55 \\
 & & & & & 33 & 37 & 43 & 48 & 51 & 57 \\
 & & & & & & 39 & 44 & 49 & 58 & 60 \\
 & & & & & & & 45 & 52 & 59 & 63 \\
 & & & & & & & & 56 & 61 & 64 \\
 & & & & & & & & & 62 & 65
\end{array}$$

and

$$H_1 = ((2,6),(1,9),(4,4),(1,4),(6,7),(1,10),(1,9),(7,8),(1,10),(10,10),(6,11)) .$$

Example 5. *Let*

$$T = \begin{array}{ccccccccccc}
19 & 9 & 64 & 7 & 32 & 54 & 65 & 13 & 61 & 50 & 46 \\
 & 1 & 2 & 4 & 8 & 11 & 14 & 16 & 21 & 23 & 37 \\
 & & 3 & 5 & 10 & 12 & 18 & 22 & 25 & 28 & 41 \\
 & & & 6 & 15 & 17 & 20 & 24 & 29 & 36 & 43 \\
 & & & & 19 & 26 & 27 & 34 & 35 & 39 & 53 \\
 & & & & & 30 & 31 & 38 & 42 & 44 & 55 \\
 & & & & & & 40 & 45 & 48 & 51 & 58 \\
 & & & & & & & 47 & 49 & 52 & 59 \\
 & & & & & & & & 56 & 57 & 62 \\
 & & & & & & & & & 60 & 63
\end{array} .$$

After the application of SPLIT 1 we obtain the pair

$$T = \begin{array}{ccccccccccc}
19 & 1 & 2 & 4 & 8 & 11 & 13 & 16 & 21 & 23 & 37 \\
 & 9 & 3 & 5 & 10 & 12 & 14 & 22 & 25 & 28 & 41 \\
 & & 64 & 6 & 15 & 17 & 18 & 24 & 29 & 36 & 43 \\
 & & & 7 & 19 & 20 & 27 & 34 & 35 & 39 & 46 \\
 & & & & 32 & 26 & 31 & 38 & 42 & 44 & 53 \\
 & & & & & 30 & 40 & 45 & 48 & 50 & 55 \\
 & & & & & & 54 & 47 & 49 & 51 & 58 \\
 & & & & & & & 65 & 52 & 57 & 59 \\
 & & & & & & & & 56 & 60 & 62 \\
 & & & & & & & & & 61 & 63
\end{array}$$

and

$$H_1 = ((1,1),(2,2),(3,3),(4,4),(5,5),(1,8),(7,7),(8,8),(5,9),(10,10),(4,11))\,.$$

Next we apply SPLIT 2. Observe that $i' = 4$ *(since* 7 *is stable) and that*

$$U = \begin{array}{ccccccccccc}
1 & 2 & 4 & 7 & 8 & 11 & 13 & 16 & 21 & 23 & 37 \\
 & 3 & 5 & 19 & 10 & 12 & 14 & 22 & 25 & 28 & 41 \\
 & & 6 & 9 & 15 & 17 & 18 & 24 & 29 & 36 & 43 \\
 & & & 64 & 19 & 20 & 27 & 34 & 35 & 39 & 46 \\
 & & & & 32 & 26 & 31 & 38 & 42 & 44 & 53 \\
 & & & & & 30 & 40 & 45 & 48 & 50 & 55 \\
 & & & & & & 54 & 47 & 49 & 51 & 58 \\
 & & & & & & & 65 & 52 & 57 & 59 \\
 & & & & & & & & 56 & 60 & 62 \\
 & & & & & & & & & 61 & 63
\end{array}\,.$$

Thus $h = 4$ *and no backward paths with respect to the* 4*-th row of a horizontal candidate strictly below the* 4*-th row ends weakly left of* 64*. Therefore we are in Case 1 of SPLIT 2. We set* $U = T$*. Since* 7 *original ended in* $(4,4)$ *we set* $H_{1,4} = (4,4)$*, since* 64 *is in* $(4,4)$ *we set* $H_{1,3} = (4,4)$*, since* 9 *is in* $(3,4)$ *we set* $H_{1,2} = (3,4)$ *and since* 19 *is in* $(2,4)$ *we set* $H_{1,1} = (2,4)$*.*

$$H_1 = ((2,4),(3,4),(4,4),(4,4),(5,5),(1,8),(7,7),(8,8),(5,9),(10,10),(4,11))$$

After the performance of SPLIT 3 we obtain the pair

$$T = \begin{array}{ccccccccccc}
1 & 2 & 4 & 7 & 8 & 11 & 13 & 16 & 21 & 23 & 37 \\
 & 3 & 5 & 9 & 10 & 12 & 14 & 22 & 25 & 28 & 41 \\
 & & 6 & 15 & 17 & 18 & 24 & 29 & 19 & 36 & 43 \\
 & & & 19 & 20 & 27 & 31 & 34 & 35 & 39 & 46 \\
 & & & & 26 & 30 & 38 & 42 & 44 & 50 & 53 \\
 & & & & & 32 & 40 & 45 & 48 & 54 & 55 \\
 & & & & & & 47 & 49 & 51 & 57 & 58 \\
 & & & & & & & 52 & 56 & 59 & 62 \\
 & & & & & & & & 60 & 61 & 64 \\
 & & & & & & & & & 63 & 65
\end{array}$$

and

$$H_1 = ((2,3),(3,9),(3,3),(5,5),(6,9),(1,8),(3,7),(9,11),(10,11),(10,10),(4,11))\,.$$

Example 6. *We consider*

$$T=\begin{array}{ccccccccccc}
26 & 4 & 13 & 30 & 47 & 24 & 58 & 65 & 53 & 25 & 60\\
 & 1 & 2 & 3 & 6 & 8 & 9 & 11 & 14 & 22 & 28\\
 & & 5 & 7 & 10 & 16 & 17 & 20 & 23 & 29 & 35\\
 & & & 12 & 15 & 18 & 19 & 21 & 32 & 40 & 43\\
 & & & & 26 & 27 & 31 & 36 & 37 & 41 & 49\\
 & & & & & 33 & 34 & 38 & 39 & 46 & 51\\
 & & & & & & 44 & 45 & 48 & 52 & 54\\
 & & & & & & & 50 & 55 & 57 & 61\\
 & & & & & & & & 56 & 59 & 62\\
 & & & & & & & & & 63 & 64
\end{array}.$$

After the application of SPLIT 1 we obtain

$$T=\begin{array}{ccccccccccc}
26 & 1 & 2 & 3 & 6 & 8 & 9 & 11 & 14 & 22 & 28\\
 & 4 & 5 & 7 & 10 & 16 & 17 & 20 & 23 & 25 & 35\\
 & & 13 & 12 & 15 & 18 & 19 & 21 & 29 & 40 & 43\\
 & & & 30 & 24 & 27 & 31 & 32 & 37 & 41 & 49\\
 & & & & 26 & 33 & 34 & 36 & 39 & 46 & 51\\
 & & & & & 47 & 38 & 45 & 48 & 52 & 54\\
 & & & & & & 44 & 50 & 53 & 57 & 60\\
 & & & & & & & 58 & 55 & 59 & 61\\
 & & & & & & & & 56 & 62 & 64\\
 & & & & & & & & & 63 & 65
\end{array}$$

and

$$H_1=((1,1),(2,2),(3,3),(4,4),(3,5),(6,6),(1,9),(8,8),(6,9),(6,10),(10,11))\,.$$

Next we apply SPLIT 2. Observe that $i'=2$ *and that*

$$U=\begin{array}{ccccccccccc}
1 & 2 & 3 & 6 & 8 & 9 & 11 & 14 & 22 & 25 & 28\\
 & 4 & 5 & 7 & 10 & 16 & 17 & 20 & 23 & 26 & 35\\
 & & 13 & 12 & 15 & 18 & 19 & 21 & 29 & 40 & 43\\
 & & & 30 & 24 & 27 & 31 & 32 & 37 & 41 & 49\\
 & & & & 26 & 33 & 34 & 36 & 39 & 46 & 51\\
 & & & & & 47 & 38 & 45 & 48 & 52 & 54\\
 & & & & & & 44 & 50 & 53 & 57 & 60\\
 & & & & & & & 58 & 55 & 59 & 61\\
 & & & & & & & & 56 & 62 & 64\\
 & & & & & & & & & 63 & 65
\end{array}.$$

Thus $h=2$ *and the backward path with respect to the* 2*-nd row of the horizontal candidate* 13 *ends weakly left of* 26. *We are in Case 2 of SPLIT 2 and set* $T=U$ *and*

$$H_1=((2,10),(2,2),(3,3),(4,4),(3,5),(6,6),(1,9),(8,8),(6,9),(6,10),(10,11))\,.$$

After the application of SPLIT 3 we finally obtain

$$T = \begin{array}{ccccccccccc}
1 & 2 & 3 & 6 & 8 & 9 & 11 & 14 & 22 & 25 & 28 \\
 & 4 & 5 & 7 & 10 & 16 & 17 & 20 & 23 & 35 & 26 \\
 & & 12 & 13 & 15 & 18 & 19 & 21 & 29 & 40 & 43 \\
 & & & 24 & 26 & 27 & 31 & 32 & 37 & 41 & 49 \\
 & & & & 30 & 33 & 34 & 36 & 39 & 46 & 51 \\
 & & & & & 38 & 44 & 45 & 48 & 52 & 54 \\
 & & & & & & 47 & 50 & 53 & 57 & 60 \\
 & & & & & & & 55 & 56 & 59 & 61 \\
 & & & & & & & & 58 & 62 & 64 \\
 & & & & & & & & & 63 & 65
\end{array}$$

and

$$H_1 = ((2,11),(3,4),(3,3),(2,4),(5,5),(1,9),(7,7),(5,8),(9,9),(6,10),(10,11))\,.$$

5. The Algorithm MERGE

Now we describe the algorithm MERGE that merges a pair of a shifted standard tableau and a shifted hook tabloid to a shifted tabloid. Again we have to introduce two routines on a partial shifted hook tabloid H of shape λ.

A reshift from cell (i,j') to cell (i,j) in a partial shifted hook tabloid. Let (i,j), (i,j') be two cells in H, $j \le j'$. We define the term *reshift from (i,j') to (i,j) in H*. The output of this operation is another shifted tabloid H' which coincides with H except for the cells (i,k), $j \le k \le j'$. Let $j < k \le j'$. If $H_{i,k-1} = (i',k-1)$ or $H_{i,k-1} = (k,j')$ then set $H'_{i,k} = (i'+1,k)$ or $H'_{i,k} = (k+1,j'+1)$, respectively. If $H_{i,k-1} = (i,l)$, $l \ge k$, then set $H'_{i,k} = (i,l)$. The cell (i,j) in H' remains empty. We denote H' by RSHIFT(H,i,j',j). For an example for a reshift from $(1,6)$ to $(1,2)$ read Figure 10 from right to left.

Observe that SHIFT and RSHIFT are inverse to each other in the following sense: Let $i \le j \le j' \le \lambda_i + i - 1$ and H be a partial shifted hook tabloid in which only cell (i,j) is empty. Then

$$H = \mathrm{RSHIFT}(\mathrm{SHIFT}(H,i,j,j'),i,j',j).$$

If H' is a partial shifted hook tabloid where only cell (i,j') is empty then

$$H' = \mathrm{SHIFT}(\mathrm{RSHIFT}(H,i,j',j),i,j,j').$$

In general the output of a reshift in a partial shifted hook tabloid need not to be a partial shifted hook tabloid for $(i'+1,k)$, respectively $(k+1,j'+1)$, need not to be a cell in the shifted Ferrers diagram of shape λ if $(i',k-1)$, respectively (k,j'), is a cell in this diagram. However, in the following two situations the reshift is indeed a partial shifted hook tabloid and we apply the reshift only if one of these situations encounters.

(1) Let T be a shifted tabloid and H an accompanying partial shifted hook tabloid. Let e be a vertical candidate in T with respect to H and the i-th row and perform reverse jeu de taquin (i being the fixed row) with it in T and with respect to a set D. Let (g,j) denote the cell of e in the output tabloid and (g',j') denote the cell of e in T. Suppose that e is the maximal vertical candidate in T under the vertical candidates in the rows h, $g \le h \le g'$, with respect to the backward paths order of T. Then the reshift from cell $(i,g'-1)$ to cell $(i,g-1)$ in H produces another partial shifted hook tabloid.

(2) Again let T be a shifted tabloid and H an accompanying partial shifted hook tabloid. Furthermore we assume in this situation that there exists no vertical candidate in T with respect to H. Let e be a horizontal candidate in T with respect to H and the i-th row and perform reverse jeu de taquin (i being the fixed row) with it in T and with respect to a set D. Let (g, j) denote the cell of e in the output tabloid and (g', j') denote the cell of e in T. Suppose that e is the smallest horizontal candidate in T under the horizontal candidates in the columns k, $j \leq k \leq j'$, with respect to the backward paths order of T. Then the reshift from cell (i, j') to cell (i, j) in H produces a partial shifted hook tabloid.

(In order to see that use Lemma 1.)

As jeu de taquin in SPLIT is mostly followed by a shift in the accompanying shifted hook tabloid, reverse jeu de taquin in MERGE is mostly followed by a reshift. Let T be a shifted tabloid, H a shifted hook tabloid, e a candidate and D a set of cells. If e is a horizontal candidate then $(T', H') = \mathrm{RJS}_D(T, H, e)$ is obtained as follows: $T' = \mathrm{RJT}_D(T, e)$, $H' = \mathrm{RSHIFT}(H, i, q, j')$ and $H'_{i,j'} = (i', j')$, where q is the column of e in T and $c_{T'}(e) = (i', j')$. If e is a vertical candidate then $(T', H') = \mathrm{RJS}_D(T, H, e)$ is obtained as follows: $T' = \mathrm{RJT}_D(T, e)$, $H' = \mathrm{RSHIFT}(H, i, p-1, i'-1)$ and $H'_{i,i'-1} = (i', j')$, where p is the row of e in T and $c_{T'}(e) = (i', j')$. Again JS and RJS are each other's respective inverse.

A retransfer from cell (i, k) to cell (i, j) in a partial shifted hook tabloid. Let $j \leq k \leq r$ and $H_{i,k} = (i', k')$ with either $i' = i$ or $k' = k$. We define the term *retransfer from cell (i, k) to cell (i, j)*. The output of this operation is again another partial shifted hook tabloid H' which coincides with H except for the cells (i, l), $j \leq l \leq k$. If $k' = k$ and $i' - i \geq k - j$ let $H'_{i,j} = (i' + j - k, j)$, otherwise $H'_{i,j} = (i, k' + i - i')$. For $j < l \leq k$ set $H'_{i,l} = (l, l)$. We denote H' by $\mathrm{RTRANS}(H, i, k, j)$. For an example of a retransfer from $(1, 6)$ to $(1, 2)$ read Figure 11 from right to left. Again whenever we apply a retransfer from (i, k) to (i, j) in H, we have $H_{i,l} = (l + 1, l + 1)$ for $j \leq l < k$ as in the example.

Observe that TRANS and RTRANS are inverse to each other in the following sense. Let $i \leq j \leq k \leq \lambda_i + i - 1$. If H is a shifted hook tabloid such that $H_{i,j}$ is in the same row or column as (i, j) and with $H_{i,l} = (l, l)$ for $j < l \leq k$ then

$$H = \mathrm{RTRANS}(\mathrm{TRANS}(H, i, j, k), i, k, j).$$

If H' is a shifted hook tabloid such that $H_{i,k}$ is in the same row or column as (i, k) and with $H_{i,l} = (l + 1, l + 1)$ for $j \leq l < k$ then

$$H' = \mathrm{TRANS}(\mathrm{RTRANS}(H', i, k, j), i, j, k).$$

Now we are in the position to state the Algorithm MERGE. Suppose we are in the course of inverting the Algorithm SPLIT in the i-th row. The 3 parts of MERGE in row i are numbered in reverse order to emphasize the connection between SPLIT i and MERGE i, $i = 1, 2, 3$.

MERGE 3. Set $z = i$. Repeat the following until there exists no vertical candidate strictly below the z-th row.

[Let e be the maximal vertical candidate strictly below the z-th row with respect to the backward paths order.

If $c_T(e) = (h,h)$: [Let h' be minimal such that $T_{j,j}$ is a vertical unstable candidate for $h' \leq j < h$. If $e = T_{h,h}$ is unstable and the backward path of smallest horizontal candidate e' strictly below the h-th row contains $(h+1,h+1)$ set $(T,H) = \mathrm{RJS}_{(h+1,h+1)}(T,H,e')$ $(*)$ and $H = \mathrm{RTRANS}(H,i,h,h'-1)$, otherwise set $H = \mathrm{RTRANS}(H,i,h,h')$. Furthermore $e = T_{h',h'}$.]

If e is a vertical candidate repeat the following:
[If $c_T(e) = (k,k)$, $k \neq z+1$, set $(T,H) = \mathrm{RJS}_{(k-1,k)}(T,H,e)$. Set $(T,H) = \mathrm{RJS}_{MD \cup z+1}(T,H,e)$.]
until either

1. e is in row $z+1$,
2. there exists a vertical candidate strictly below the row of e,
3. $c_T(e) = (l,l)$ and the backward path of a horizontal candidate includes $(l+1,l+1)$,
4. $c_T(e) = (l,l)$ and $T_{l-1,l-1}$ is an unstable vertical candidate.

If e is in row $z+1$ set $z = z+1$.]

Observe that if we are in the if-condition (If $c_T(e) = (h,h)$: $[\ldots]$) at the beginning of a step of MERGE 3, then there exists no vertical candidate strictly below the h-th row.

MERGE 2. Let i'' be such that for $i+1 \leq h \leq i''$ there exists a vertical candidate entry in row h and all other candidates are horizontal.

We distinguish between two cases. In Case 2 we consider the case when the smallest horizontal candidate strictly below the i''-th row is smaller than the smallest vertical candidate.

Case 1. [Set a =True. Let $H_{i,i''} = (h,k)$. We define $T_{i,k-h+i}$ to be a vertical candidate in row i. Repeat for $g = i$ to $g = i''$: [If a =True let e be the vertical candidate in row g or $g+1$ that is maximal with respect to the backward paths order. If $a =$ False let e be the vertical candidate in row g. Let $T = \mathrm{RJT}_{(g,g)}(T,e)$. If the backward path includes $(g+1,g+1)$ set a =False. If e was the candidate in the $(g+1)$-st row and $T_{g,k}$ is the vertical candidate in row g, then let $H_{i,g} = (g+1,k+1)$.] For $i \leq g \leq i''$, let $H_{i,g} = (g,g)$.]

Case 2. [Repeat for $g = i$ to $g = i''-1$: [Let e be the entry in cell $H_{i,g}$ of T and set $T = \mathrm{RJT}_{MD}(T,e)$.] Let $i \leq i' \leq i''$, be such that in the previous step no entry moved to (i',i'). Set $H = \mathrm{RTRANS}(H,i,i'',i')$. For every g, $i \leq g < i'$, let $H_{i,g} = (g,g)$.]

MERGE 1. Repeat for $j = i$ to $j = \lambda_i + i - 1$: [Let e be the minimal horizontal candidate with respect to the backward paths order in T. Set $(T,H) = \mathrm{RJS}_{(i,j)}(T,H,e)$ and $H_{i,j} = \emptyset$.]

Let T be a shifted tabloid which is ordered up to (i,i) and H a partial shifted hook tabloid, where no entry in the i-th row is empty. Let (T',H') denote the pair we obtain after the application of MERGE 3, MERGE 2 and MERGE 1 to the i-th row of (T,H). We denote $T' = \mathrm{STAND\_MERGE}(T,H,i)$ and note that H' is equal to H with the entries in the i-th row deleted. We invite the reader to apply MERGE 3, MERGE 2 and MERGE 1 to the first rows of the output pairs in the examples in Section 4 in order to find out that this application results in the input pairs.

We are finally in the position to formulate the Algorithm MERGE. The input is a shifted standard tableaux T of shape λ and a shifted hook tabloid H of shape λ.

MERGE. Set $T' = T$ and $H' = H$. Repeat for $i = 1$ down to $i = r$: [$T' =$ HOOK_MERGE(T', H', i) and delete the i-th row of H'.]

The output tabloid T' of the algorithm is denoted by STAND_MERGE(T, H). Note that the output tabloid H' is empty.

In the remaining section we aim to show the following theorem.

Theorem 3. *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be a partition with distinct components and $1 \leq i \leq r$.*

(1) *Let T be a shifted tabloid of shape λ which is ordered up to $(i+1, i+1)$ and H a partial shifted hook tabloid of shape λ, where the cells of the i-th row are empty. Let T' = STAND_SPLIT(T, H, i) and H' = HOOK_SPLIT(T, H, i). Then T = STAND_MERGE(T', H', i).*
(2) *Let T be a shifted tabloid of shape λ which is ordered up to (i, i) and H a partial shifted hook tabloid of shape λ, where no cell in the i-th row is empty. Let T' = STAND_MERGE(T, H, i) and H' denote the partial shifted hook tabloid H with the entries in the i-th row deleted. Then T = STAND_SPLIT(T', H', i) and H = HOOK_SPLIT(T', H', i).*

This theorem, once it is proved, shows that SPLIT is the desired algorithm, i.e. a bijection from T_λ onto $S_\lambda \times H_\lambda$, for MERGE is its inverse.

In the following we fix a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ with distinct components and a row i, $1 \leq i \leq r$.

6. The Proof

6.1. Variantes of SPLIT 1 and SPLIT 2

We modify SPLIT 1 and SPLIT 2 such that after this modification MERGE i is the inverse of SPLIT i, $i = 1, 2, 3$. The original version of SPLIT and the modification of SPLIT, where we replace SPLIT i, $i = 1, 2$, by their modifications described below, are equivalent, since we only change the order of some commuting steps.

We start with a modification of SPLIT 2. Consider the following Algorithm POST-SPLIT 2, which we apply to a pair to which we have applied SPLIT 2.

POST-SPLIT 2. We continue with Case i, $i=1,2$, if we were in Case i in SPLIT 2.

Case 1. [If $T_{i',i'}$ is a vertical candidate, let i'', $i' \leq i''$, be maximal such that for $i' \leq j < i''$, $T_{j,j}$ is unstable and $T_{j+1,j+1}$ is a horizontal candidate and set $H = \mathrm{TRANS}(H, i, i', i'')$. Otherwise omit this step and set $i'' = i'$.]

Case 2. [If $T_{i'+1,i'+1}$ is a horizontal candidate, let i'', $i'+1 \leq i''$, be maximal such that for $i'+1 \leq j \leq i''$, $T_{j,j}$ is unstable, $T_{j+1,j+1}$ is a horizontal candidate and set $H = \mathrm{TRANS}(H, i, i', i'')$. Furthermore set $(T,H) = \mathrm{JS}_{(i''+1,k)}(T, H, T_{i''+1,i''+1})$, where $(i''+1, k)$ was the last cell in the forward path of $T_{i''+1,i''+1}$ in the $(i''+1)$-st row. Otherwise omit this step and set $i'' = i'$.]

In both cases: $U = T$. If we were in Case 2 set $U = \mathrm{JT}_{i''+2}(U, T_{i''+1,k})$. Repeat for $g = i''$ down to $g = i'+1$: [Let e be the vertical candidate in row g of T and set $U = \mathrm{JT}_{g+1}(U, e)$.]

Repeat for $g = i''$ down to $g = i'+1$: [Let e be the vertical candidate in row g of T and set $(T,H) = \mathrm{JS}_{(g,l)}(T, H, e)$, where (g,l) was the last cell in the forward path of e in the g-th row in the shifted tabloid U in the previous step. (Note that the forward path of e ends in (g,l).)]

Again the following holds for (T,H) after the application of POST-SPLIT 2: The smallest horizontal candidate strictly below the i''-th row is smaller than every vertical candidate if and only if we were in Case 2 of POST-SPLIT 2.

Furthermore observe that the application of SPLIT 2 and SPLIT 3 is equivalent to the application of SPLIT 2, POST-SPLIT 2 and SPLIT 3, for POST-SPLIT 2 is roughly speaking just the beginning of SPLIT 3.

Touching from above. Let (T,H) denote a pair to which we apply SPLIT 2 and SPLIT 3. Assume that we are in Case 2 in a certain step of SPLIT 3. Then we say *e touches e' from above* and for $h < j \leq h'$, $T_{j-1,j-1}$ touches $T_{j,j}$ from above. If e is a horizontal candidate before the application of Case 2 we say that $(h-1, h-1)$ is the *place of change for e* and (j,j) is the place of change for $T_{j,j}$, $h \leq j < h'$. If e is already a vertical candidate before the application of Case 2 we say that (j,j) is the place of change for $T_{j,j}$, $h \leq j \leq h'$. Furthermore: If we are in Case 1 of SPLIT 2 let $T_{g,g}$, $i \leq g \leq i''$ be the exceptional entries, where T denotes the shifted tabloid at the beginning of SPLIT 2. If we are in Case 2 of SPLIT 2 let $T_{g,g}$, $i \leq g \leq i''$ and $g \neq i', i'+1$, be the exceptional entries, T again being the shifted tabloid at the beginning of SPLIT 2. If $i' \neq i''$ and we are in Case 2 of SPLIT 2 then the vertical candidate in row $i'+1$ after the application of SPLIT 2 is defined to be an entry which touches from above with place of change $(i'+1, i'+1)$.

We replace SPLIT 1 by the following variante. Again the input is a shifted tabloid T, which is ordered up to $(i+1, i+1)$ and H a partial shifted hook tabloid, where the i-th row is empty.

Variante of SPLIT 1. Repeat for $j = \lambda_i + i - 1$ down to $j = i$: We set $(T,H) = \mathrm{JS}_D(T, H, T_{i,j})$, where the set D is defined as follows. If $T_{i,j}$ is exceptional or touched from above let $D = MD$. If $T_{i,j}$ is neither exceptional nor touched from above but touches from above, let $D = \{\rho\}$, where ρ is the place of change of $T_{i,j}$. In all other cases $D = \emptyset$.]

The change of SPLIT 1 forces us to replace SPLIT 2 by the following slight modification. This is because for a pair (T,H) which falls into Case 2 of the original version of SPLIT 2, the entry $T_{i',i'}$ moves either to a stable position or to its place of change if we apply the Variante of SPLIT 1.

> **Variante of SPLIT 2.** Choose i', $i \leq i' \leq r$, maximal such that $H_{i,k} = (k,k)$ and $T_{k-1,k-1}$ is unstable (i.e. $T_{k-1,k-1} > T_{k-1,k}$) for $i < k \leq i'$.
> Set $U = T$. Repeat for $g = i'$ down to $g = i$: Set $U = \mathrm{JT}_{g+1}(U, T_{g,g})$.
>
> We distinguish between two cases. Let h be minimal, $i \leq h \leq i'$, such that $T_{h,h}$ is not in the $(h+1)$-st row of U. We continue with Case 2 below if h does not exist or $T_{i',i'}$ is stable and the backward path with respect to the i'-th row of U of a horizontal candidate in row $i'+1$ or below ends weakly to the left of the cell of $T_{i'-1,i'-1}$ in the i'-st row. In all other cases we continue with Case 1.
>
> Reject the tabloid U we constructed so far in this step.
>
> *Case 1.* Repeat for $g = i'$ down to $g = h+1$: [Let k be such that either (g,k) or $(g+1,k)$ is the endcell of the forward path of $T_{g,g}$ in the procedure for constructing U. Set $T = \mathrm{JT}_{(g,k)}(T, T_{g,g})$ and $H_{i,g-1} = (g,k)$.]
> Let $(h,k) = \mathrm{CJT}(T, T_{h,h})$ and set $T = \mathrm{JT}(T, T_{h,h})$. If $h - k \leq i - i'$ let $H_{i,i'} = (i, i-h+k)$ otherwise let $H_{i,i'} = (i'+h-k, i')$.
> Repeat for $g = h-1$ down to $g = i$: [Let $H_{i,g} = \mathrm{CJT}_{g+1}(T, T_{g,g})$ and $T = \mathrm{JT}_{g+1}(T, T_{g,g})$.]
> *Case 2.* If $T_{i',i'}$ is unstable, set $i' = i'+1$. Repeat for $g = i'-1$ down to $g = i$: [Let $H_{i,g} = \mathrm{CJT}_{g+1}(T, T_{g,g})$ and $T = \mathrm{JT}_{g+1}(T, T_{g,g})$.]
> Now we apply POST-SPLIT 2.

Check that if we replace SPLIT 1 and SPLIT 2 in SPLIT by the variantes above, this variante of SPLIT is equivalent to the original version. In the following SPLIT denotes this variante and SPLIT i, $i = 1,2,3$, its parts. Furthermore SPLIT' and SPLIT' i, $i = 1,2,3$, denote the versions that were valid before this paragraph. Note that after the application of SPLIT 2 the following holds: If we were in Case 1 and $T_{i''+1,i''+1}$ is a horizontal candidate then $T_{i'',i''}$ is a stable vertical candidate, if we were in Case 2 and $T_{i''+1,i''+1}$ and $T_{i''+2,i''+2}$ are horizontal candidates then $T_{i''+1,i''+1}$ is stable.

In Section 3, before SPLIT 3 we define a set C, which we need in order to apply SPLIT 3. Note that we can omit the candidates in C that neither touch nor are touched from above in the course of SPLIT 3 and consequently these entries are omitted in the following in C.

6.2. The main lemmas I

The lemmas in this section we need in order to prove that SPLIT 1 and MERGE 1 are inverse to each other. They have analogs that are used for showing that SPLIT 3 and MERGE 3 are each other's respective inverse as we see in Subsection 6.4.

We introduce some notation concerning the relative position of an entry and a path. Let e and e' be two entries in a shifted tabloid T in the i-th row and $c_T(e) = (i,j)$, $c_T(e') = (i,k)$. If $j < k$, resp. $j \leq k$, we say that e' is right, resp. weakly right, of e. If P is a path in the shifted tabloid T then e' is said to be (weakly) right of P, if P includes an entry e such that e' is (weakly) right of e. Similar definitions apply to left, weakly left, above, weakly above, below and weakly below. If the formulations of our lemmas and corollaries include phrases in brackets $[\dots]$, the assertions are true with and without these phrases.

In the following lemma we characterize the largest and the smallest entry with respect to the backward paths order of a given set of entries. See Figure 14.

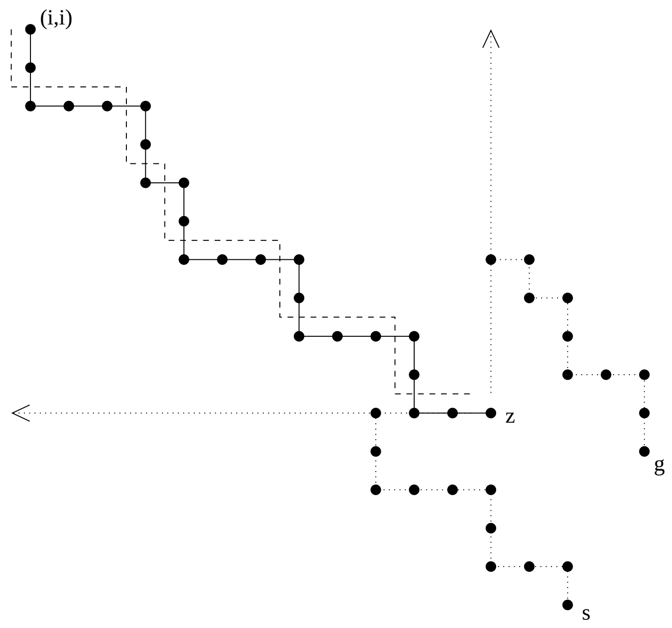


FIGURE 14. The full line indicates the backward path of entry z, the circles indicate the centers of the traversed cells. The dashed line indicates the border: The entries greater than z with respect to the backward path order are located north-east of this border and the entries smaller than Z are located south-west of this border. Moreover observe that g is greater than z for its backward path (dotted line) enters the column of z weakly above of z and s is smaller than z for its backward path enters the row of z weakly left of z.

Lemma 1. *Let Z be a set of entries in a shifted tabloid T.*

(1) *Let $s \in Z$ and P_s denote the backward path of s. Then s is the smallest entry in Z with respect to the backward paths order, if and only if every $z \in Z$ is either [weakly right and] above of P_s or the backward path of z enters the column of s weakly above of s.*

(2) *Let $g \in Z$ and P_g denote the backward path of g. Then g is the greatest entry in Z with respect to the backward paths order, if and only if every $z \in Z$ is either left [and weakly below] of P_g or the backward path of z enters the row of g weakly left of g.*

We extend the definition of 'entry e touches entry e' from above': Let T be a shifted tabloid, e, e' two entries in T and D a set of cells in the shifted Ferrers diagram. Let P be the forward path of e with respect to D and Q the backward path of e' in T. If there exists a cell (k, k) on the main diagonal such that $(k-1, k-1), (k-1, k) \in P$ and $(k, k) \in Q$, we say that e *touches* e' *from above with respect to* D. If we want to refer to the restricted definition for touching from above which was valid before this paragraph we add the phrase 'in the application of SPLIT'. Note that there exist pairs (T, H) in which an entry e touches another entry e' from above in the sense we just defined, but e does not touch e' from above in the application of SPLIT to (T, H) in the sense we defined in Subsection 6.1.

The following lemma is fundamental. It is illustrated in Figure 15.

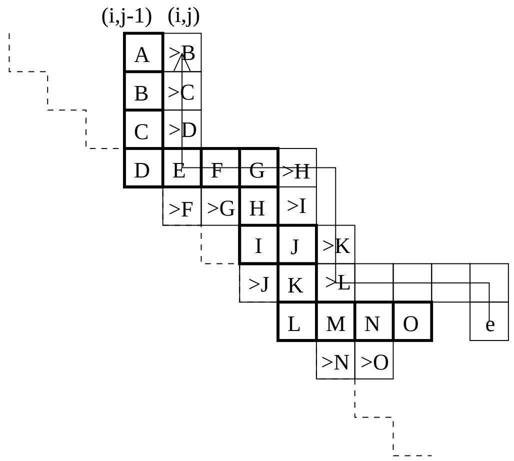


FIGURE 15. $P_{i,j-1} = \{A, B, C, \dots, M, N, O\}$. Observe that the entry to the right of 'A' is greater than the entry in the cell labeled with 'B' in T (therefore labelled with '$> B$') for $P_{i,j-1}$ starts with a south step. The corresponding assertion holds for the other labels because of the run of $P_{i,j-1}$. Furthermore a possible backward path Q (full line) of e is drawn. Consider for example the cell of Q labelled with 'E'. Then, by reverse jeu de taquin, the next cell of Q must be the cell labelled with '$> D$' and not the cell labelled with 'D'.

Lemma 2. *Let $(i, j-1)$ and (i, j) be two cells in a shifted tabloid T, which is ordered up to (i, j). Let $P_{i,j-1}$ denote the forward path of $T_{i,j-1}$ in T with respect to the a set D and e' the entry of the endcell of $P_{i,j-1}$ in T. Let e be an entry in T whose backward path (i being the fixed row) contains (i, j). If $T_{i,j-1}$ does not touch e from above with respect to D, then e is right [and weakly above] of $P_{i,j-1}$ or the backward path of e enters the column of e' weakly above of e'. (See Figure 16.)*

Proof. If $P_{i,j-1}$ consists solely of south steps there is nothing to prove. Thus we assume that there is at least one east step in $P_{i,j-1}$.

We show the following: Let e'' be below of $P_{i,j-1}$. If $T_{i,j-1}$ does not touch e'' from above with respect to D then every cell in the backward path P of e'' (i being the fixed row) in a column greater than $j-1$ is below of $P_{i,j-1}$.

If the statment were false there would exist g,l such that $(g, l), (g, l+1) \in P_{i,j-1}$ and $(g, l+1), (g+1, l+1) \in P$. Since $T_{i,j-1}$ does not touch e'' from above with respect to D the cell (g, l) is not a cell in the main diagonal and thus $(g+1, l)$ is a cell in the shifted Ferrers diagram. From $(g, l), (g, l+1) \in P_{i,j-1}$ it follows that $T_{g,l+1} < T_{g+1,l}$, which is a contradiction to $T_{g,l+1} > T_{g+1,l}$, which follows from $(g, l+1), (g+1, l+1) \in P$.

If the statement in the lemma were false, e is below of $P_{i,j-1}$ or the backward path of e enters the column of e' below of e'. By the assertion we have just proved every entry in the backward path P of e which is in a column greater than $j-1$ and weakly left of the column of e' must be below of $P_{i,j-1}$. But, since (i, j) is contained in the backward path of e and (i, j) is not below of $P_{i,j-1}$, this is a contradiction if $P_{i,j-1}$ does not consist solely of south steps. $\square$

Note that if we are in the situation of Lemma 2 and e enters the column of e' weakly above of e' then e enters the column of e' weakly above the cell with the smallest row in $P_{i,j-1}$ in the column of e' by the proof of the lemma.

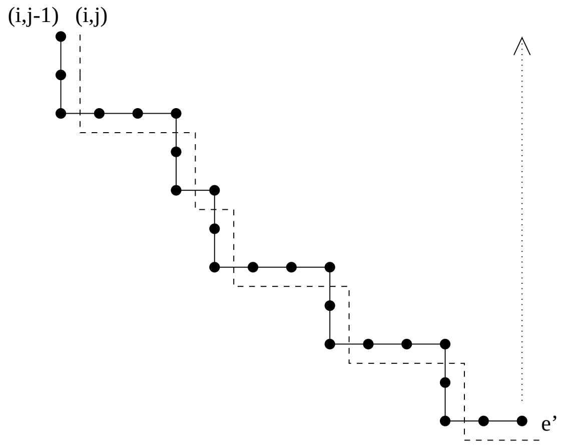


FIGURE 16. The full line is the forward path of $T_{i,j-1}$ in T, the circles indicate the center of the cells the forward path traverses. The dashed line is the border, possible cells for e are located north-east of this border.

Remark 1. *Mostly we apply the lemma in the following situation. Let $(i, j-1)$ and (i, j) be two cells in a shifted tabloid T which is ordered up to the predecessor of (i, j) in the total order. Furthermore let $P_{i,j}$ denote the forward path of $T_{i,j}$ in T with respect to a set D and let $P_{i,j-1}$ denote the forward path of $T_{i,j-1}$ in T' with respect to a set D', where T' denotes the shifted tabloid we obtain after performing jeu de taquin in T with $T_{i,j}$ and with respect to D. Let e' denote the entry in the last cell of $P_{i,j-1}$ in T'. Observe that $P_{i,j}$ coincides with the backward path of $T_{i,j}$ in T' (i being the fixed row) except for some parts of the i-th row. Thus, if $T_{i,j-1}$ does not touch $T_{i,j}$ from above with respect to D' in T', then $T_{i,j}$ is right [and weakly above] of $P_{i,j-1}$ in T' or the backward path of $T_{i,j}$ in T' enters the column of e' weakly above of e', by Lemma 2.*

Observe that Figure 14 and Figure 16 coincide after a shift of the (dashed) domain border in Figure 14 by the vector $(1, -1)$. This leads us to the following.

Corollary 2. *Let $(i, j-1)$, (i, j), T and T' be as in Remark 1. Furthermore let Z' be a set of cells in T' in the last $r-i+1$ rows, including the cell of $T_{i,j}$, such that $T_{i,j}$ is the smallest entry under the entries in the cells of Z' with respect to the backward paths order in T'. Let T'' denote the shifted tabloid we obtain after performing jeu de taquin with $T_{i,j-1}$ in T' and with respect to D' and let j' denote the column of $T_{i,j-1}$ in T''. Let Z'' denote the set of cells we obtain from Z' by first replacing every cell (h, k) with $k \le j'$ by $(h-1, k-1)$ and then deleting the cells in the $(i-1)$-st row. If $T_{i,j-1}$ touches no entry in a cell in Z' from above with respect to D' in T' then $T_{i,j-1}$ is smaller than every entry in a cell in Z'' with respect to the backward paths order in T''.*

Proof. By Lemma 1 (1), Lemma 2 and Remark 1 it is clear that either the entry in the cell of Z'' that came from $T_{i,j}$ is greater than $T_{i,j-1}$ with respect to the backward paths order in T'' or the cell that came from $T_{i,j}$ was deleted in the course of constructing Z''. (This is due to the following observation: Let ρ be a cell which is right and weakly above of a path P then $\rho - (1, 1)$ is weakly right and above of P or P has no cell in the row of $\rho - (1, 1)$.)

Now suppose that e is an entry in T' weakly below the i-th row that it is greater than $T_{i,j}$ with respect to the backward paths order in T'. Thus (note the relative position of e and $T_{i,j}$ by Lemma 1) and since the backward path (i being the fixed row) of $T_{i,j}$ in T' includes (i, j), the backward path of e in T' includes (i,j). (Observe that the backward path of e coincides with the backward path of $T_{i,j}$ after they meet.) By Lemma 2 this implies that e is either right [and weakly above] of $P_{i,j-1}$ in T' or the backward path of e enters the column of the end e' of $P_{i,j-1}$ in T' weakly above of e'. From this the assertion follows by Lemma 1 (1). □

In Section 3 after SPLIT' 1 we claim that for an entry e whose forward path terminates in a diagonal cell (k,k) in the course of SPLIT' 1, we have $T_{k,k} = e$ and $H_{i,k} = (k,k)$ after the application of SPLIT' 1. This is also true for SPLIT 1.

Lemma 3. *Let T be a shifted tabloid which is ordered up to $(i+1, i+1)$ and H a partial shifted hook tabloid where the i-th row is empty. Let (T^j, H^j) denote the pair we obtain in the course of applying SPLIT 1 to (T,H) in the i-th row after performing jeu de taquin with $T_{i,j}$. Assume that the forward path of $T_{i,j}$ in T^{j+1} contains cells on the main diagonal and let (k,k) be the first of these in the run of the forward path. Let $h < j$ be such that the forward path P of $T_{i,h}$ in T^{h+1} ends weakly right of the k-th column. Furthermore suppose that the forward path of $T_{i,l}$ in T^{l+1} does not contain a cell on the main diagonal for $h < l < j$. Then $T_{i,h}$ touches $T_{i,j}$ in T^{h+1} from above, to be more accurate $(k-1,k-1),(k-1,k) \in P$ and (k,k) is the cell of $T_{i,j}$ in T^{h+1}.*

Proof. We show that the backward path of $T_{i,j}$ in T^{h+1} does not contain a cell (g,g) on the main diagonal with $i < g < k$. Furthermore this backward path contains $(i, h+1)$ and (k,k). The assertion is then a consequence of Lemma 2: Let T in Lemma 2 equal T^{h+1}, j equal $h+1$ and let $D = \{(g,k)\}$, where (g,k) is the first cell in the forward path P of $T_{i,h}$ in T^{h+1} in column k. By Lemma 2 $T_{i,j}$ is either right of P or the backward path of $T_{i,j}$ enters column k weakly above of (g,k) or $T_{i,h}$ touches $T_{i,j}$ from above in T^{h+1}. Since $g < k$ the first two options are impossible and thus $T_{i,h}$ touches $T_{i,j}$ from above. The only possible place for touching is $(k-1,k-1)$, $(k-1,k)$, (k,k) for the backward path of $T_{i,j}$ in T^{h+1} does not contain a cell on the main diagonal strictly between the i-th and the k-th column.

Observe that the backward path (i being the fixed row) of $T_{i,j}$ in T^j contains no diagonal cell (g,g) with $i < g < k$ and it contains (i,j) and therefore $(i,h+1)$, for it coincides with the forward path of $T_{i,j}$ in T^{j+1} except for parts of the i-th row. If $j-1 = h$ there is nothing to prove. Otherwise the forward path of $T_{i,j-1}$ in T^j contains no cell on the main diagonal and thus the backward path of $T_{i,j}$ in T^{j-1} does not contain a cell on the main diagonal strictly between column i and k and it contains $(i,j-1)$, for this backward path coincides with the backward path of $T_{i,j}$ in T^j until it meets the backward path of $T_{i,j-1}$ in T^{j-1} and after that it coincides with the backward path of $T_{i,j-1}$ in T^{j-1}. The claim at the beginning of the proof now follows by induction with respect to $j-h$. □

Now we are ready to show the assertion before the formulation of Lemma 3: Assume that the forward path of $T_{i,k}$ in T^{k+1} contains a cell ρ on the main diagonal and let $h < k$ be such that the forward path of $T_{i,h}$ in T^{h+1} ends weakly right of the column of ρ. Let D be the appropriate set of cells in SPLIT 1 such that $T^h = \mathrm{JT}_D(T^{h+1}, T_{i,h})$, then it suffices to show that the forward path of $T_{i,h}$ in T^{h+1} with respect to D ends strictly left of the column of ρ. We show the assertion with induction with respect to $k-h$. To this end let j, $j > h$, be minimal such that the forward path of $T_{i,j}$ in T^{j+1} with respect to the appropriate set in SPLIT 1 contains a cell ρ' on the main diagonal. By Lemma 3 the forward path of $T_{i,h}$ in T^{h+1} with respect to D ends in a column strictly left of the column of ρ' and the column of ρ' is smaller than the column of ρ by induction since $k-j < k-h$.

Actually the corollary also implies the following inversion of the statement before: If $H_{i,k} = (k,k)$ after the application of SPLIT 1, then $T_{k,k}$ is an entry with which we perform jeu de taquin in the course of applying SPLIT 1 and (k,k) is the end of its forward path.

In fact this is a consequence of the following observation which can be shown by Lemma 2 and Lemma 3: Assume that the backward path of a horizontal candidate in (T^j, H^j) contains (k,k). Then there exists an l, $j \leq l \leq \lambda_i + i - 1$ such that the forward path of $T_{i,l}$ in T^{l+1} with respect to the appropriate D in SPLIT 1 contains (k,k) and $T_{i,l}$ is the smallest horizontal candidate in (T^j, H^j), whose backward path contains (k,k).

6.3. SPLIT 1 and MERGE 1 are each other's respective inverse

We define property AFTER_SPLIT 1 for a pair (T,H) and will observe that the application of SPLIT 1 yields a pair with that property. A pair (T,H) of a shifted tabloid T and a partial shifted hook tabloid H has property AFTER_SPLIT 1 if

(1) the last $r - i + 1$ rows of H form a shifted hook tabloid,
(2) there exists no vertical candidate (with respect to the i-th row) in (T,H),
(3) the subtabloid of T consisting of the last $r-i+1$ rows is standard, except for $T_{k,k} \in C$ there might hold $T_{k,k} \geq T_{k,k+1}$, where C is the set of entries $T_{k,k}$ with $H_{i,k} = (k,k)$.
(4) $T_{k,k} \in C$ such that $T_{k,k}$ either proves to be exceptional or touched from above in the application of SPLIT, there exists a horizontal candidate e in T whose backward path includes (j,j) and $e <_T T_{k,k}$, for every diagonal cell $(j,j) \neq (k,k)$ in the backward path of $T_{k,k}$, and
(5) $T_{k,k} \in C$ is unstable and neither exceptional nor touched from above in the application of SPLIT, then $T_{k,k}$ touches another entry from above in the application of SPLIT and is in its place of change.

Claim 1.

(1) Let T be a shifted tabloid which is ordered up to $(i+1, i+1)$ and H a partial shifted tabloid such that the i-th row is empty. Let (T', H') denote the pair we obtain after the application of SPLIT 1. Then (T', H') has property AFTER_SPLIT 1 and the application of MERGE 1 to (T', H') yields (T,H).
(2) Let (T', H') be a pair with property AFTER_SPLIT 1 and (T,H) denote the pair we obtain after the application of MERGE 1. Then T is ordered up to $(i+1, i+1)$ and H is a partial shifted hook tabloid such that the i-th row is empty and the application of SPLIT 1 to (T,H) yields (T', H').

Proof.
re 1. Let (T^j, H^j) denote the pair we obtain after performing jeu de taquin with $T_{i,j}$ and the corresponding shift in the partial shifted hook tabloid in SPLIT 1 and set $(T^{\lambda_i+i}, H^{\lambda_i+i}) = (T,H)$. By construction the current cell of $T_{i,j+1}$ in T^{j+1} is in $\{H^{j+1}_{i,j+1}, H^{j+1}_{i,j+2}, \ldots, H^{j+1}_{i,\lambda_i+i-1}\}$. Furthermore by induction we assume that $T_{i,j+1}$ is the smallest horizontal candidate in a cell in $\{H^{j+1}_{i,j+1}, H^{j+1}_{i,j+2}, \ldots, H^{j+1}_{i,\lambda_i+i-1}\}$ with respect to the backward paths order in T^{j+1}. Let D be the set such that T^j is obtained from T^{j+1} in SPLIT 1 by performing jeu de taquin with $T_{i,j}$ and with respect to D. By construction no horizontal candidate in a cell in $\{H^{j+1}_{i,j+1}, H^{j+1}_{i,j+2}, \ldots, H^{j+1}_{i,\lambda_i+i-1}\}$ is touched from above by $T_{i,j}$ with respect to D in T^{j+1}: Let (k,k) be the cell on the main diagonal with k is minimal such that the backward path in T^{j+1} of a horizontal candidate with respect to H^{j+1} contains (k,k). Then the forward path of $T_{i,j}$ with respect to D ends strictly left of the column of k by Lemma 3. (See also the last paragraph in the previous subsection.) Thus, by Corollary 2, $T_{i,j}$ is smaller or equal

than every horizontal candidate in a cell in $\{H^j_{i,j}, H^j_{i,j+1}, \dots, H^j_{i,\lambda_i+i-1}\}$ with respect to the backward paths order in T^j. Therefore if we apply the $j-i+1$-st step of MERGE 1 to the pair (T^j, H^j) we reobtain (T^{j+1}, H^{j+1}).

re 2. We define the following algorithm PRE-MERGE 1. The input is a pair (T', H') with the property AFTER_SPLIT 1.

> **PRE-MERGE 1.** Repeat the following: [Let (j,j) be such that $H'_{i,j} \neq (j,j)$ but the backward path in T' of a horizontal candidate contains (j,j). If such a j does not exist then stop. Otherwise let e be the smallest horizontal candidate with that property and set $(T', H') = \mathrm{RJS}_{(j,j)}(T', H', e)$.]

Observe that the application of PRE-MERGE 1 and MERGE 1 to (T', H') yields the same result as the application of MERGE 1 to (T', H') for the smallest horizontal candidate whose backward path contains (j,j) is the smallest horizontal candidate with respect to the backward paths order weakly left of the j-th column. Let (T'', H'') denote the pair we obtain after application of PRE-MERGE 1 to (T', H'). Then (T'', H'') has the properties 1 – 3 from AFTER_SPLIT and for every cell (k,k) on the main diagonal such that the backward path of a horizontal candidate contains (k,k), $T''_{k,k}$ is itself a horizontal candidate.

We denote the property of (T'', H'') by AFTER_SPLIT' 1. (If we apply SPLIT' 1 to a pair (T, H), T ordered up to $(i+1, i+1)$ and the i-th row of H is empty, then the output pair has property AFTER_SPLIT' 1.)

We show the following: Let (T, H) denote the pair we obtain after the application of MERGE 1 to (T'', H''). If we apply SPLIT' 1 to (T, H) we reobtain (T'', H'').

Let (T''^j, H''^j) denote the pair we obtain in the course of performing MERGE 1 to (T'', H'') after performing reverse jeu de taquin with respect to $\{(i, j-1)\}$ and the corresponding reshift in the partial shifted hook tabloid, and set $(T''^i, H''^i) = (T'', H'')$. By induction the backward path of every horizontal candidate in a cell in $\{H''^j_{i,j}, H''^j_{i,j+1}, \dots, H''^j_{i,\lambda_i+i-1}\}$ in T''^j contains (i,j). Let e be the smallest horizontal candidate in a cell in $\{H''^j_{i,j}, H''^j_{i,j+1}, \dots, H''^j_{i,\lambda_i+i-1}\}$ with respect to the backward paths order in T''^j. Furthermore let P be its backward paths in T''^j with respect to (i,j). Observe that P does not contain a cell on the main diagonal except for possibly $c_{T''^j}(e)$ and (i,j). If j' is the column of e in T''^j, the subtabloid of T''^j consisting of the first j' columns is ordered up to (i,j) by property AFTER_SPLIT' 1 and the minimality of e. Thus $e = T''^{j+1}_{i,j}$ and performing jeu de taquin with e in T''^{j+1} and with respect to the cells on the main diagonal together with the corresponding shift in H''^{j+1} results in the pair (T''^j, H''^j). By the minimality of e, Lemma 1 (1) and by the construction of the reshift in H''^j the entries in the cells in $\{H''^{j+1}_{i,j+1}, H''^{j+1}_{i,j+2}, \dots, H''^{j+1}_{i,\lambda_i+i-1}\}$ are right [and weakly above] of P or their backward paths in T''^{j+1} enter column j' weakly above of $c_{T''^j}(e)$. Thus, by the argument from Lemma 2, the backward paths of the entries in these cells in T''^{j+1} contain $(i, j+1)$.

We define another Algorithm POST-SPLIT 1 which we apply to a pair (T'', H'') with property AFTER_SPLIT' 1.

> **POST-SPLIT 1.** Repeat the following: [Let $T''_{j,j}$ be an unstable horizontal candidate which is neither exceptional nor touched from above and not already in its place of change, if $T''_{(j,j)}$ has a place of change. If $T''_{j,j}$ does not exist stop. Otherwise: If $T''_{j,j}$ has a place of change ρ let $D = \{\rho\}$ otherwise let $D = \emptyset$. Set $(T'', H'') = \mathrm{JS}_D(T'', H'', T''_{j,j})$.]

Now the assertion follows since PRE-MERGE 1 and POST-SPLIT 1 are inverse to each other (in order to show that use Lemma 3) and since the application of SPLIT' 1 and POST-SPLIT 1 is equivalent to the application of SPLIT 1. □

In view of Claim 1 it remains to show the following:

(1) Let (T, H) be a pair with the property AFTER_SPLIT 1 and let (T', H') denote the pair we obtain after the application of SPLIT 2 and SPLIT 3 to (T, H). We have to show that T' is ordered up to (i, i), the subtabloid of H' consisting of the last $r-i+1$ rows is a shifted hook tabloid and if we apply MERGE 3 and MERGE 2 to (T', H') we reobtain (T, H).
(2) Let (T, H) be such that T is ordered up to (i, i) and the subtabloid of H consisting of the last $r-i+1$ rows is a shifted hook tabloid. Let (T', H') denote the pair we obtain after the application of MERGE 3 and MERGE 2. We have to show that (T', H') has property AFTER_SPLIT 1 and that the application of SPLIT 2 and SPLIT 3 to (T', H') yields (T, H).

6.4. The main lemmas II

The following lemma is similar to Lemma 2.

Lemma 4. *Let (i, m) and $(i-1, n)$, $m \leq n$, be two cells in a shifted tabloid T such that the subtabloid of T consisting of the last $r-i+1$ rows and without the entries in (i, m) and $(i-1, n)$ has increasing rows and columns. Let $P_{i-1,n}$ denote the forward path of $T_{i-1,n}$ in T' with respect to a set, where T' denotes the shifted tabloid we obtain after performing jeu de taquin in T with $T_{i,m}$ and with respect to D. Furthermore let e denote the last entry of $P_{i-1,n}$ in T'. Then $T_{i,m}$ is [weakly left and] below of $P'_{i-1,n} = P_{i-1,n} \cup \{(i-1, m), (i-1, m+1), \ldots, (i-1, n-1)\}$ in T' or the backward path of $T_{i,m}$ with respect to (i, m) in T' enters the row of e weakly left of e.*

Proof. Similar to the proof of Lemma 2. Observe that if the backward path of $T_{i,m}$ in T' enters the row of e weakly left of e then it enters the row of e weakly left of the leftmost cell of $P_{i-1,n}$ in the row of e. □

Corollary 3. *Let $(i-1, n)$, (i, m), T, T' be as in Lemma 2. Furthermore let Z' be a set of cells weakly below the i-th row, which includes the cell of $T_{i,m}$ in T', such that $T_{i,m}$ is the greatest entry in a cell of Z' in T' with respect to the backward paths order. Let T'' denote the shifted tabloid we obtain after performing jeu de taquin with $T_{i-1,n}$ in T' and with respect to a set and let i' denote the row of $T_{i-1,n}$ in T''. Construct a set Z'' of cells as follows: Replace every cell (h, k) in Z' with $h \leq i'$ by $(h-1, k-1)$. Then $T_{i-1,n}$ is greater than every entry in a cell of Z'' in T'' with respect to the backward paths order.*

Proof. Similar to the proof of Corollary 2. □

6.5. SPLIT 2 and MERGE 2 are each other's respective inverse

We define property AFTER_SPLIT 2 for a pair (T, H) and will observe that a pair to which we have applied SPLIT 1 and SPLIT 2 has that property. A pair (T, H) of a shifted tabloid T and a partial shifted hook tabloid H has property AFTER_SPLIT 2 if

(1) the last $r-i+1$ rows of H form a shifted hook tabloid,
(2) there exists an i'' such that for every $i+1 \leq g \leq i''$ there is a vertical candidate in row g. Besides these no other vertical candidates exist. For $i+1 \leq g < g+1 \leq i''$ the vertical candidate in row g is greater than the vertical candidate in row $g+1$ with

respect to the backward paths order (i being the fixed row). If we perform jeu de taquin with the vertical candidates in row $i'', i''-1, \ldots, i+1$ in that order, then the cell of the vertical candidate in row g in T is the last cell in the g-th row of the forward path of the candidate. Moreover: $T_{i'',i''}$ is a stable vertical candidate or $T_{i''+1,i''+1}$ is a stable horizontal candidate or $T_{i''+1,i''+1}$ is not a horizontal candidate or $T_{i''+2,i''+2}$ is not a horizontal candidate,

(3) the subtabolid of T consisting of the last $r-i+1$ rows is standard, except for $T_{p,q} \in C' \cup C$ there might hold $T_{p,q} > \min(T_{p+1,q}, T_{p,q+1})$, where C' and C are the two sets we define before the application of SPLIT 3 to (T, H) (see Section 3 before the description of SPLIT 3),

(4) e is a horizontal candidate which is touched from above in the application of SPLIT, then there exists a horizontal candidate e' with $e' <_T e$ and the backward path of e' contains (j,j), for every diagonal cell $(j,j) \neq c_T(e)$ with $i'' < j$ in the backward path of e, and

(5) e is an unstable horizontal candidate which is not touched from above in the application of SPLIT then e touches from above in the application of SPLIT and is in its place of change.

A remark on Property (2) of AFTER_SPLIT 2: Note that the vertical candidate in row g is greater than the vertical candidate in row $g+1$, $i+1 \leq g < g+1 \leq i''$ in a pair to which we have applied SPLIT 2 by Lemma 4.

Claim 2.

(1) Let (T, H) be a pair with property AFTER_SPLIT 1 and let (T', H') denote the pair we obtain after the application of SPLIT 2. Then (T', H') has property AFTER_SPLIT 2 and the application of MERGE 2 to (T', H') results in (T, H).

(2) Let (T', H') be a pair with property AFTER_SPLIT 2 and (T, H) denote the pair we obtain after the application of MERGE 2. Then (T, H) has property AFTER_SPLIT 1 and the application of SPLIT 2 to (T, H) results in (T', H').

Proof. Left to the reader. Observe that we are in Case 2 of SPLIT 2 respectively Case 2 of MERGE 2 if and only if the smallest horizontal candidate with respect to the backward paths order in (T', H') in a row strictly below the lowest vertical candidate is smaller than the smallest vertical candidate. The rest follows from the fact that jeu de taquin and reverse jeu de taquin are inverse to each other and from the fact that (appart from one exception) the endcells of the forward paths are not distorted by a shift in the accompanying shifted hook tabloid. The exception is the exceptional vertical candidate in the h-th row in Case 1 of SPLIT 2, respectively in the i-th row in Case 1 of MERGE 3, for which we need Corollary 3. □

6.6. Irreducible pairs

Touching from below. We define an analog to 'e touches e' from above' for the algorithm MERGE. Suppose we are at the beginning of a step of MERGE 3 and the if-condition (If $c_T(e) = (h,h) : [\ldots]$) is true. In the following commands in the description of MERGE 3 we distinguish between two cases. If we are in the first case we say that e' touches e from below and for $h' < j \leq h$, $T_{j,j}$ touches $T_{j-1,j-1}$ from below. Furthermore for $h' \leq j \leq h$, let (j,j) be the place of change for $T_{j,j}$. In the second case we say that for $h' < j \leq h$, $T_{j,j}$ touches $T_{j-1,j-1}$ from below and (j,j) is the place of change for $T_{j,j}$.

We define the exceptional entries with respect to MERGE 2. Suppose we have applied MERGE 2, Case 1. Define $T_{g,g}$ to be exceptional for $i \leq g \leq i''$. Now suppose we have applied MERGE 2, Case 2. For $i \leq g \leq i''$ and $g \neq i', i'+1$ we define $T_{g,g}$ to be exceptional and $T_{i'+1,i'+1}$ is said to be touched from below by the smallest candidate strictly below the $(i'+1)$-st row if $i' \neq i''$.

A *step of SPLIT 3* includes the choice of a protagonist $e \in C'$ and the performance of either Case 1, Case 2 or Case 3. A *step of MERGE 3* is defined analogously. The maximal vertical candidate e at the beginning of a step of MERGE 3 is said to be the first protagonist of the step and the candidate e after the performance of the if-condition (If $c_T(e) = (h,h) : [\dots]$) is said to be the second protagonist. Clearly the second protagonist can be equal to the first protagonist in a step of MERGE 3.

We first show that SPLIT 3 and MERGE 3 are inverse to each other for a certain class of pairs (T,H).

Irreducible pair with respect to SPLIT. Let (T,H) be a pair with property AFTER_SPLIT 2. Suppose that in the course of applying SPLIT 3 to (T,H) every protagonist which touches an entry from above was either in C' at the beginning of SPLIT 3 or is itself touched by an entry from above in the application of SPLIT. Observe that this is equivalent to the fact that we are never in Case 3 in SPLIT 3 and that $C' \neq \emptyset$ at the beginning of every step of SPLIT 3. Then (T,H) is said to be irreducible with respect to SPLIT. In Figure 17 the application of SPLIT 3 to an irreducible pair is illustrated.

Irreducible pair with respect to MERGE. Let T be a shifted tabloid which is ordered up to (i,i) and H a partial shifted hook tabloid, such that the subtabloid consisting of the last $r-i+1$ rows is a shifted hook tabloid. If every protagonist e (except for possibly the last) in MERGE 3 either touches an entry from below or terminates in row $z+1$ in a certain step of MERGE 3, then (T,H) is said to be irreducible with respect to MERGE. Note that this is equivalent to the following: Suppose the second protagonist in a step of MERGE 3 is a horizontal candidate. Then this is either the last step in the application of MERGE 3 to (T,H) or this second protagonist is equal to e' in the if-condition at the beginning of the next step of MERGE 3.

Claim 3. Let (T,H) be irreducible with respect to SPLIT and let (T',H') denote the pair we obtain after the application of SPLIT 3. Then T' is ordered up to (i,i), the last $r-i+1$ rows of H' form a shifted hook tabloid and the application of MERGE 3 to (T',H') results in (T,H).

Proof. Let C'_{start} denote the set C' at the beginning of the application of SPLIT 3 to (T,H). Suppose we are in the course of applying SPLIT 3: We choose in a certain step entry e_1 to be the protagonist and obtain the pair (U,J), in the next step we choose e_2 to be the protagonist and obtain (U',J'). Let z' be such that there exists a vertical candidate in row z' of (U,J) and in C'_{start}, which is not the protagonist in and before the step of e_1 and if there exists a vertical candidate in the $(z'+1)$-st row of (U,J) and in C'_{start}, this entry was the protagonist in or before the step of e_1. If z' does not exist, set $z'=i$. Let z'' be the corresponding quantity for the step of e_2 and observe that $z''=z'-1$ iff e_2 is the vertical candidate of (U,J) in the z'-th row. Otherwise $z''=z'$. If we are in Case 2 in the step of e_i let h_i and h'_i denote h and h' in the description of Case 2 of SPLIT 3, $i=1,2$.

By induction we assume the following: Suppose we are in Case 1 in the step of e_1. Then e_1 is greater or equal than every vertical candidate strictly below the z'-th row of (U,J) with respect to the backward paths order and if e_1 is horizontal candidate then e_2 is the greatest vertical candidate strictly below the z''-th row of (U,J). Now suppose we are in Case 2 in the step of e_1. Then $e_2 = U_{h'_1,h'_1}$ is greater or equal than every vertical candidate strictly

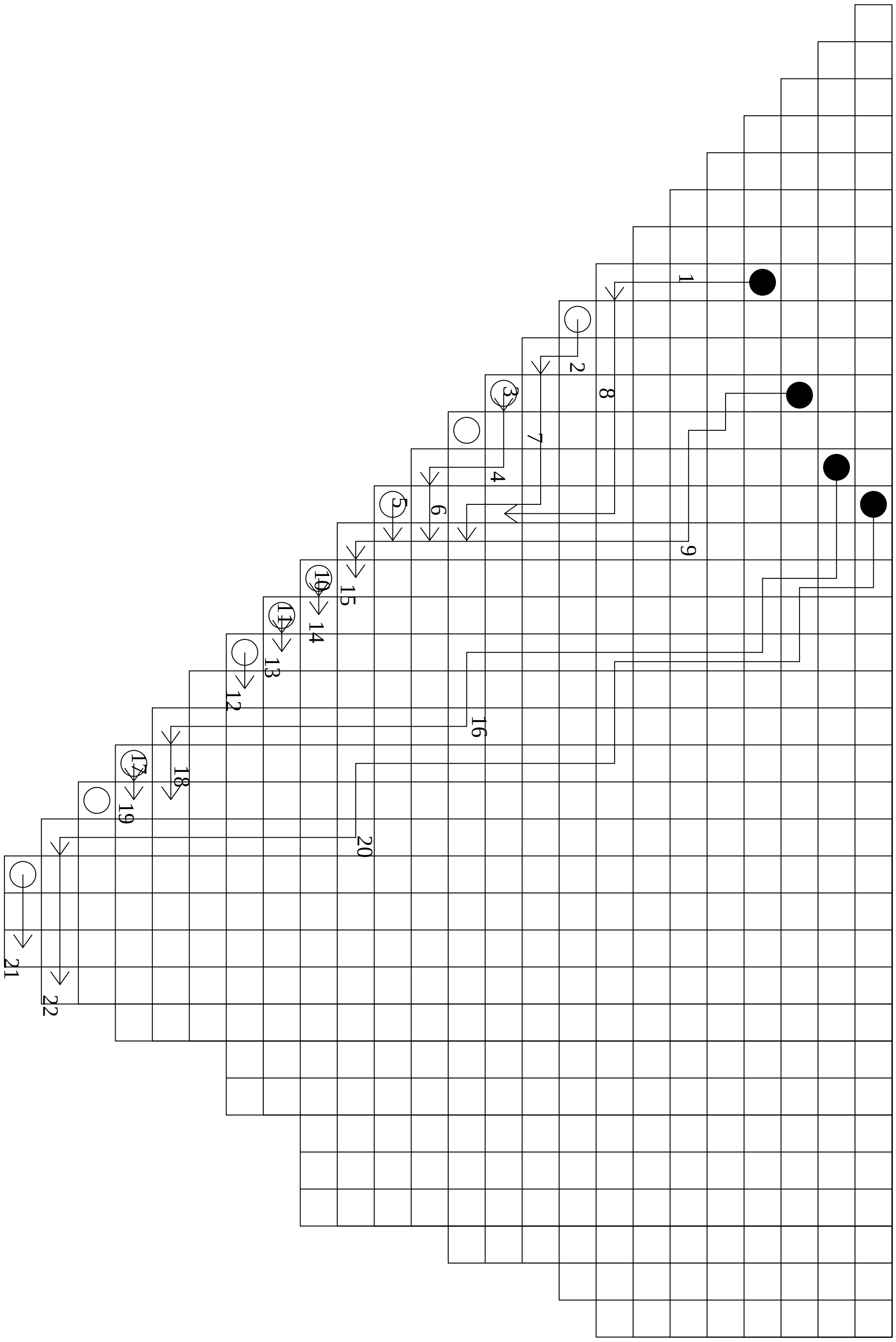


FIGURE 17. The application of SPLIT 3 to a irreducible pair. A full circle indicates a candidate in C', an empty circle indicates a candidate in C at the beginning of the application. The lines are the forward paths of jeu de taquin, the numbers near them indicate their order in the application of SPLIT 3. If the tip of an arrow touchs the border of a cell, this indicates that the entry in the cell is noticed to be unstable at that point in the application of the algorithm.

below the z'-th row of (U,J) and if e_2 is a horizontal candidate in (U,J) then $U_{h'_1-1,h'_1-1}$ is the greatest vertical candidate strictly below the z'-th row of (U,J).

We aim to show that the application of one step of MERGE 3 to (U',J') with $z=z''$ either yields

(1) (U,J) or
(2) the pair we obtain before performing TRANS in the step of e_1 (in which we are in Case 1 of SPLIT 3 and e_1 is a horizontal candidate at the beginning of the step) or
(3) a pair which results in (U,J) after the application of a second step of MERGE 3 until $(*)$ (see the description of MERGE 3).

Furthermore we show the following (induction step): Suppose we are in Case 1 in the step of e_2. Then e_2 is greater or equal than every vertical candidate in (U',J') strictly below the z''-th row and if e_2 is a horizontal candidate then $e_2=U_{k,k}$ for a k and $U'_{k-1,k-1}=U_{k-1,k-1}$ is the greatest vertical candidate of (U',J') strictly below the z''-th row. Now suppose we are in Case 2 in the step of e_2 . In this case $U'_{h'_2,h'_2}$ is greater or equal then every vertical candidate of (U',J') strictly below the z''-th row of (U',J') and if $U'_{h'_2,h'_2}$ is a horizontal candidate then $U'_{h'_2-1,h'_2-1}$ is the greatest vertical candidate of (U',J') strictly below the z''-th row.

We distinguish between the two cases according to whether or not we are in Case 1 in the step of e_1.

Case 1. There are two cases. Either

(1) $e_1,e_2\in C'_{start}$, e_2 is in the z'-th row of U and e_1 was either the vertical candidate of C'_{start} in the $(z'+1)$-st row or the unique horizontal candidate of C'_{start} in (T,H) or
(2) $c_U(e_2)=(k,k)$ for a k, e_2 is unstable in U and the backward path of e_1 in U contains $(k+1,k+1)$.

re 1. Clearly e_2 is the greatest vertical candidate in (U,J) weakly below the z'-th row for e_1 is greater or equal than every vertical candidate in (U,J) strictly below the z'-th row by induction and e_2 is greater than e_1 by property AFTER_SPLIT 2. If e_2 changes row in the step of e_2, the first cell in row $z'+1$ in its forward path is weakly right of the first cell in row $z'+1$ in (U,J) in the backward path of e_1 by property AFTER_SPLIT 2. Thus e_2 is also the greatest vertical candidate in (U',J') strictly below the z''-th row by Corollary 3, if we are not in Case 2 in the step of e_2 and in this case it is obvious that e_2 is the first and the second protagonist in the application of a step of MERGE 3 to (U',J') with $z=z''$ (check that the if-condition at the beginning of the step of MERGE 3 is either not fulfilled or its application leaves (U',J') unchanged in this step). Otherwise $U'_{h'_2,h'_2}$ is the greatest vertical candidate in (U',J') strictly below the z''-th row for firstly $e_2=U'_{h_2-1,h_2-1}$ is the greatest vertical candidate strictly below the z''-th row and strictly above the h_2-th row of (U',J') by Corollary 3, secondly there exists no vertical candidate strictly below the h'_2-th row and thirdly $U'_{h'_2,h'_2}$ is greater than $U'_{h,h}$ for $h_2-1\le h<h'_2$. Thus $U'_{h'_2,h'_2}$ is the first protagonist in the application of a step of MERGE 3 to (U',J'). Consequently e_2 is the second protagonist in the application of a step of MERGE 3 to (U',J') if we are in Case 2 in the step of e_2. It remains to show that the loop in the application of a step of MERGE 3 with $z=z''$ to (U',J') terminates with '1. e is in row $z+1$': Observe that the forward path of e_2 in U does not contain a cell (k,k) such that there exists a vertical candidate strictly below the k-th row for such a vertical candidate would be greater than e_2 in U by the arguments from Lemma 2, which is a contradiction to the maximality of e_2 weakly below the z'-th row in (U,J). Thus the loop does not stop with '2.'. Furthermore the forward path of e_2 in (U,J)

with respect to the appropriate set does not contain a cell (k,k) (except for possibly the endcell of the forward path of e_2 with respect of the emptyset) with $h_2 - 1 \neq k$ such that the backward path in U of a horizontal candidate contains $(k+1,k+1)$, for otherwise e_2 touches the minimal horizontal candidate whose backward paths contains $(k+1,k+1)$ from above in the application of SPLIT by Lemma 3, which implies $J_{i,k+1} = (k+1,k+1)$ and this is a contradiction to the choice of h_2. Therefore the loop does not stop with '3.'. Finally, if k is the row of e_2 in U', then the subtabloid of U consisting of the cells strictly below of the z'-th row and weakly above of the k-th row is standard and consequently the loop does not stop with '4.'. Therefore the application of one step of MERGE 3 to (U',J') with $z = z''$ yields (U,J) in this case.

re 2. There are two cases: either e_1 is a vertical candidate in (U,J) or e_1 is a horizontal candidate in (U,J).

Let e_1 be a vertical candidate, then e_1 is the greatest vertical candidate strictly below the z'-th row of (U,J) by the induction hypothesis. If we are not in Case 2 in the step of e_2 then e_2 is the greatest vertical candidate strictly below the z''-th row of (U',J') by Corollary 3 (since e_2 is unstable in (U,J)) and thus the first and the second protagonist in the application of a step of MERGE 3 to (U',J') with $z = z''$. If we are in Case 2 in the step of e_2 then again $U'_{h'_2,h'_2}$ is the greatest vertical candidate strictly below the z''-th row of (U',J') and with this the first protagonist in the application of a step of MERGE 3 to (U,J). Thus e_2 is the second protagonist in the application of a step of MERGE 3 to (U',J') with $z = z''$ in this case. It remains to show that the loop at the end of the application of a step of MERGE 3 to (U',J') stops with '2.' and at the cell of e_2 in U. This is left to the reader, for it is similar to the previous case.

Now let e_1 be a horizontal candidate, then e_1 is greater than every vertical candidate strictly below the z'-th row of (U,J) and e_2 is the greatest vertical candidate strictly below the z'-th row of (U,J) by the induction hypothesis. Thus either e_2 (if we are in Case 1 in the step of e_2) or $U'_{h'_2,h'_2}$ (if we are in Case 2 in the step of e_2) is the greatest vertical candidate strictly below the z''-th row of (U',J') and therefore the application of a step of MERGE 3 to (U',J') with $z = z''$ yields (U,J). Note that the loop in MERGE 3 terminates with '3.' and at the cell of e_2 in U.

Case 2. Clearly $e_2 = U_{h'_1,h'_1}$ and e_2 is greater or equal than every vertical candidate strictly below the $z' = z''$-th row of (U,J) with respect to the backward paths order by the induction hypothesis. Thus either e_2 (if we are in Case 1 in the step of e_2) or $U'_{h'_2,h'_2}$ (if we are in Case 2 in the step of e_2) is greater or equal than every vertical candidate strictly below the z''-th row.

Suppose e_2 is a vertical candidate in (U,J). Then either e_2 or $U'_{h'_2,h'_2}$ is the first protagonist in the application of a step of MERGE 3 with $z = z''$ to (U',J') and therefore e_2 is the second protagonist. Thus the application of a step of MERGE 3 yields (U,J). Observe that the loop in MERGE 3 terminates with '4.' and at the cell of e_2 in U.

Suppose e_2 is a horizontal candidate in (U,J). First we suppose that we are in Case 1 in the step of e_2. Then $U'_{h'_1-1,h'_1-1}$ is the greatest vertical candidate strictly below the z''-th row of (U',J') and consequently the first protagonist in the application of a step of MERGE 3 to (U,J). Thus $e' = e_2$ in the if-condition at the beginning of this step of MERGE 3 and e_1 is the second (horizontal) protagonist. We obtain the pair which we had immediately before the performance of TRANS in the step of e_1. Now suppose we are in Case 2 in the step of e_2. Clearly $U'_{h'_2-1,h'_2-1}$ is the greatest vertical candidate strictly below the z''-th row of (U',J')

and consequently the first protagonist in the application of a step of MERGE 3 to (U', J') with $z = z''$. The second protagonist is the horizontal candidate e_2 and since $U'_{h'_1-1,h'_1-1}$ is the first protagonist in the next step of MERGE 3 we have $e' = e_2$ in the if-condition at the beginning of this next step. Thus the application of another step of MERGE 3 until $(*)$ yields (U, J). $\square$

Claim 4. Let (T, H) be irreducible with respect to MERGE and (T', H') denote the pair we obtain after the application of MERGE 3 to (T, H). Then (T', H') has property AFTER_SPLIT 2 and the application of SPLIT 3 to (T', H') yields (T, H).

Proof. In the course of applying MERGE 3 to (T, H) we construct two sets C' and C. We start with $C = C' = \emptyset$. If the if-condition (If $c_T(e) = (h, h)$: $[\dots]$) at the beginning of a step of MERGE 3 is true and we have just applied the appropriate commands, we set $C = C \cup \{T_{h'+1,h'+1}, T_{h'+2,h'+2}, \dots, T_{h+1,h+1}\}$ and $C' = C' \setminus \{T_{h'+1,h'+1}, T_{h'+2,h'+2}, \dots, T_{h+1,h+1}\}$ if we are in the first case in these commands, and we set $C = C \cup \{T_{h'+1,h'+1}, T_{h'+2,h'+2}, \dots, T_{h,h}\}$ and $C' = C' \setminus \{T_{h'+1,h'+1}, T_{h'+2,h'+2}, \dots, T_{h,h}\}$ if we are in the second case in these commands. After the if-condition and its commands we let $C' = C' \cup \{e\}$, whether or not the if-condition was fulfilled.

Suppose that after a certain step in the application of MERGE 3 to (T, H) we obtain the pair (U', J'), the sets C' and C and a row z. First we show that the second protagonist e of this step is also the protagonist in the application of one step of SPLIT 3 to (U', J') with the sets C' and C. This is equivalent to the fact that e is the candidate in C' whose row is maximal.

If the second protagonist is a horizontal candidate in a certain step of MERGE 3, then it is the horizontal candidate e' in the if-condition at the beginning of the next step (if we are not in the final step) of MERGE 3 and thus leaves C' at that point, for (T, H) is irreducible with respect to MERGE 3. Consequently C' includes at most one horizontal candidate and if this is the case then all other candidates in C' are strictly above the row of the unique horizontal candidate. (In order to see that note that whenever $c_T(e) = (h, h)$ at the beginning of a step of MERGE 3 then there exists no vertical candidate strictly below the h-th row by the maximality of e.) Therefore it suffices to show the assertion for second protagonists e that are vertical candidates.

In a step of MERGE 3 the second protagonist either terminates in row $z + 1$ or in a cell (k, k) on the main diagonal, if the protagonist is a vertical candidate. In the latter case by the maximality of the second protagonist (observe that after the if-condition in a step of MERGE 3 the second protagonist is greater or equal than every vertical candidate strictly below the z-th row; this is because the backward path of the first protagonist always contains the cell of the second protagonist at the beginning of a step of MERGE 3), every backward path of a vertical candidate strictly below the k-th row in the tabloid we obtain after this step contains $(k+1, k+1)$ by the argument in the proof of Lemma 2 and is therefore greater than e with respect to the backward paths order. However, the vertical second protagonist is still greater than every vertical candidate strictly below the z-th row and strictly above the k-th row, since this part of the shifted hook tabloid remains unchanged in this step of MERGE 3. Consequently in the course of MERGE 3 the second vertical protagonists (and with this the elements of C') are arranged in increasing rows (as long as the first protagonist is equal to the second protagonist) until from time to time a former second vertical protagonist leaves C' after the performance of the commands in the if-condition. We conclude that the second protagonist in a step of MERGE 3 either starts (by its maximality and the arguments in this paragraph) and ends (by the stopping criteria '4. $c_T(e) = (l, l)$ and $T_{l-1,l-1}$ is an unstable vertical candidate' in MERGE 3) below the lowest vertical candidate in C' or is equal to the lowest vertical candidate in C', whose backward path in this step ends below the second lowest vertical candidate in C' (if it exists) by the fourth stopping criteria. Consequently the row of the vertical second protagonist in a step of MERGE 3 is always below the other vertical candidates in C' and by the observation about the horizontal candidates in C' this proves our claim in the second paragraph of the proof.

Let (U, J) denote the pair before the step of MERGE 3, in which e is the second protagonist and let C'_{before} and C_{before} denote the accompanying sets. We have to show that if we apply one or two steps of SPLIT 3 to (U', J') with the sets C' and C, we reobtain (U, J) together with C'_{before} and C_{before} or a pair of a shifted tabloid and a shifted hook tabloid together with the two sets which we obtained in the application of MERGE 3 to (T, H) before the pair (U, J). Proving this claim is now a matter of considering every circumstance in the step of e in MERGE 3. (Case distinction: Firstly suppose that we are in the first case in the if-condition at the beginning of the step of MERGE 3 (this leaves no choice open for the rest of the step) and secondly combine the second case in the if-condition and the case that the if-condition is false with each of the four stopping criterions of the loop in MERGE 3.) $\square$

Thus we have shown that for irreducible pairs SPLIT 3 and MERGE 3 are inverse to each other. Now we show the assertion for reducible pairs by induction with respect to the number of protagonists not contained in C' at the beginning of the application of SPLIT 3 which touch from above but are not touched from above in the application of SPLIT, respectively with respect to the number of second protagonists unequal to the last second protagonist which neither touch an entry from below nor terminate in row $z+1$ in the application of MERGE 3.

6.7. Reducible pairs

Observe that the pair in Example 3 in Section 4 is reducible. In our proof of Claim 5 we sometimes refer to that example. Moreover in Figure 18 the application of SPLIT 3 to a reducible pair is illustrated.

For the proof of Claim 5 we need the following observation: Suppose we are in Case 3 of SPLIT 3 and let (U, J) denote the current pair before the application of Case 3. Then the protagonist e in the description of Case 3 of SPLIT 3 is a vertical candidate. In order to show that suppose that e was a horizontal candidate. Then we are either in the first step of SPLIT 3 or e joined C' in the previous step of SPLIT 3 in which we were in Case 2, where e equals $T_{h',h'}$ in the description of Case 2. We only consider the latter case here, for the first case is similar. By the relative position of e and e' in the current step of SPLIT 3 and the argument from Lemma 2 the backward path of e' in (U, J) contains $(h'+1, h'+1)$. By Lemma 3 e touches the smallest horizontal candidate e'' in (U, J) whose backward path in U contains $(h'+1, h'+1)$ from above in the application of SPLIT 3 and this is a contradiction, since e'' is weakly above of the row of e' by its minimality and the fact that e' is on the main diagonal.

Claim 5. Let (T, H) be a pair with property AFTER_SPLIT 2 and (T', H') the pair we obtain after the application of SPLIT 3. Then T' is ordered up to (i, i), the subtabloid of H' consisting of the last $r-i+1$ rows is a shifted hook tabloid and the application of MERGE 3 to (T', H') yields (T, H).

Proof. Let C'_{start} and C_{start} denote C' and C at the beginning of the application of SPLIT 3 to (T, H). We assume that every entry in C_{start} is either touched from above or touches an entry from above in the application of SPLIT 3. By Claim 3 we furthermore assume that (T, H) is reducible with respect to SPLIT.

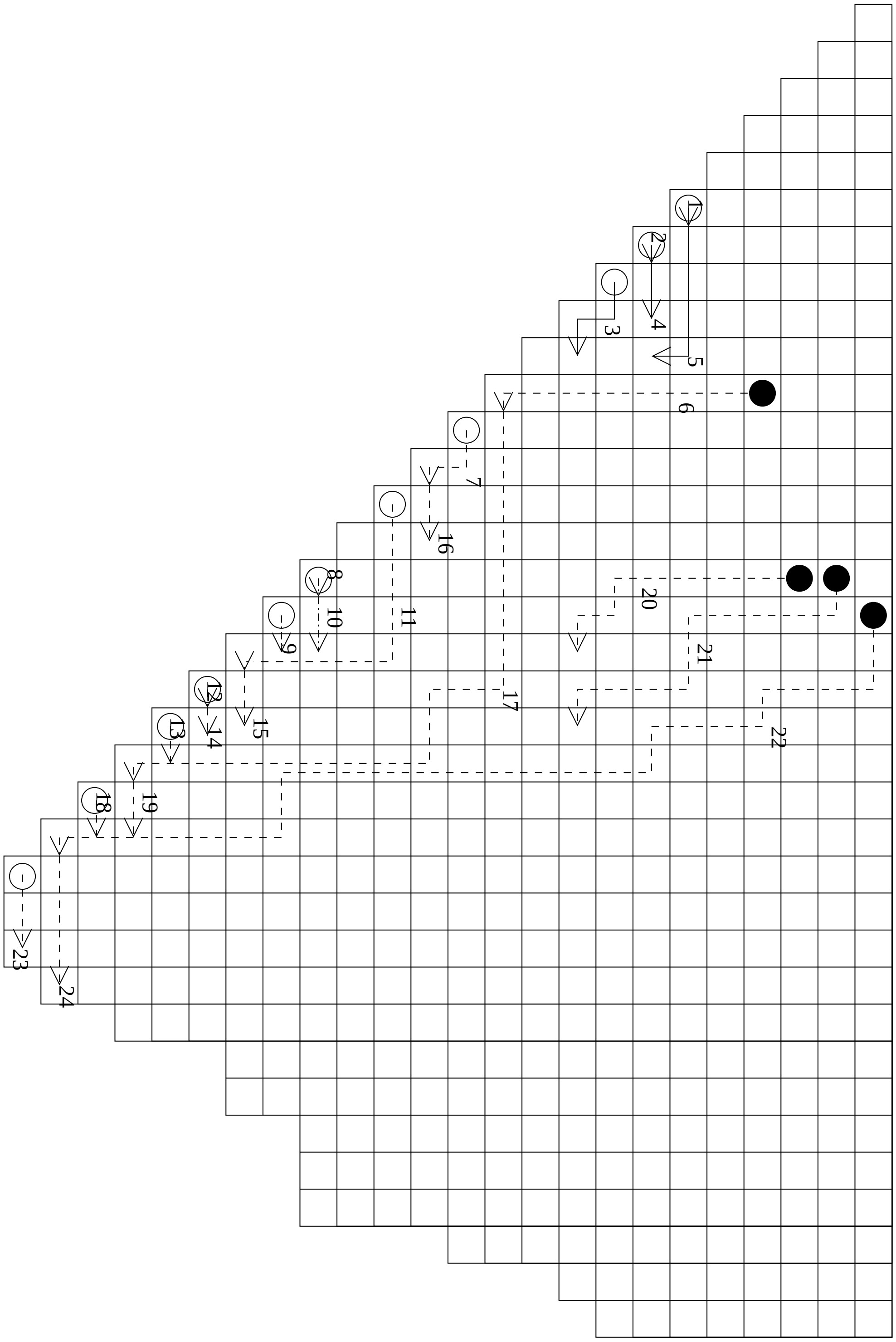


FIGURE 18. The application of SPLIT 3 to a reducible pair. (See caption of Figure 17 for an explanation.) Observe that for distinct irreducible 'parts' different line styles are used.

Let e be the candidate in C_{start} with maximal row k, which is not touched from above in the application of SPLIT ($e = 49$ in Example 3). Furthermore, if it exists, let $e' \in C_{start}$ be the entry with minimal row strictly below the k-th row such that e' is touched from above in a certain step in the course of SPLIT 3 by a protagonist which is at the beginning of the step in a row strictly above of the k-th row ($e' = 63$ in the example). Let C_1 denote the entries in C_{start} which are strictly above the k-th row and weakly below the row of e' and C_2 denote the entries in C_{start} which are strictly below the k-th row and strictly above the row of e' in T.

Let SPLIT 3.1 denote the application of SPLIT 3 with $C' = C'_{start}$ and $C = C_1$ to (T, H) and let SPLIT 3.2. denote the application of SPLIT 3 with $C' = \{e\}$ and $C = C_2$ to (T, H). Observe that the application of SPLIT 3 with $C' = C'_{start}$ and $C = C_{start}$ to (T, H) is equivalent to the simultaneous application of SPLIT 3.1 and SPLIT 3.2, where after every step of SPLIT 3.i, $i = 1, 2$, we can decide to make either the next step of SPLIT 3.1 or the next step of SPLIT 3.2, but whenever a protagonist in SPLIT 3.1 is moved weakly below the k-th row SPLIT 3.2 has already terminated. This is because of the maximality of k and because of the fact that at some step of SPLIT 3 we are either in Case 3 with e being the e' in the description of SPLIT 3 or $C' = \emptyset$ at the beginning of the step and e is the candidate with minimal row in C.

Observe that after the termination of SPLIT 3.2 and before a protagonist of SPLIT 3.1 moves weakly below the k-th row, e is the greatest vertical candidate under the vertical candidates that arose in the application of SPLIT 3.2 by the proof of Claim 3, for e is the final protagonist in SPLIT 3.2 and k is maximal. If the first protagonist of SPLIT 3.1 moves weakly below of the k-th row, then this protagonist is a vertical candidate (since whenever we are in Case 3 in SPLIT 3 the protagonist is a vertical candidate; this is proved at the beginning of this subsection) and greater than e with respect to the backward paths order by Corollary 3. Moreover every subsequent protagonist of SPLIT 3.1, which moves weakly below the k-th row, is greater than the previous. Thus the vertical candidates from SPLIT 3.1 dominate the vertical candidates from SPLIT 3.2 once they come weakly below k-th row.

Let V_1 denote the vertical pointers that came from SPLIT 3.1 and let V_2 denote the vertical pointers that came from SPLIT 3.2 after the application of SPLIT. By the induction hypothesis we reobtain (T, H) if we apply MERGE 3 to the pointers in V_1 (denoted by MERGE 3.1) and, seperately, to the pointers in V_2 (denoted by MERGE 3.2) such that when we start applying it to the pointers originated in V_2 the lowest vertical pointers that came from V_1 is strictly above the k-th row. If we apply MERGE 3 not seperately to the vertical pointers in V_1 and V_2, but to all vertical pointers in (T', H) at once, then the first steps of this application are equal to the first steps of MERGE 3.1 until the vertical pointers of MERGE 3.1 have moved strictly above the k-th row, for the candidates originated in V_1 dominate the candidates originated in V_2 with respect to the backward paths order until they are strictly above the k-th row as we saw in the pervious paragraph. $\square$

Claim 6. Let (T, H) be such that T is ordered up to (i, i), the subtabloid of H consisting of the last $r - i + 1$ rows is a shifted hook tabloid and let (T', H') denote the pair we obtain after the application of MERGE 3. Then (T', H') has property AFTER_SPLIT 2 and the application of SPLIT 3 to (T', H') yields (T, H).

Proof. By Claim 4 we assume that (T, H) is reducible with respect to MERGE 3. Let e be the protagonist in the application of MERGE 3 to (T, H) which neither touches an entry from below nor terminates in row $z + 1$, whose row k of the place of change is maximal. We bipartite the vertical pointers of (T, H) into two sets V_1 and V_2. The set V_2 includes the vertical pointer in (T,H) that results in e and if $p_1 \in V_2$ and the vertical pointer p_2 results in a protagonist that touches the protagonist that corresponds to p_1 from below then $p_2 \in V_2$. All other vertical pointers are in V_1. Observe that by the algorithm a candidate originated in V_2 can be a protagonist only after the candidates originated in V_1 have moved strictly above the k-th row: By the proof of Claim 4 a candidate e' originated in V_2 and unequal to e can only be a protagonist if e was already a protagonist and has moved to a row above of e'. Therefore we only have to show that e is a protagonist only after the pointers originated in V_1 have moved strictly above the k-th row. But this is obvious since if e is a protagonist in a step of MERGE 3 then every vertical candidate weakly below the k-th row is originated in V_2. (After e is protagonist for the first time and until it changes into a horizontal candidate, e is greater than every vertical candidate strictly above of the current row of e and strictly below the z-th row. Moreover if there exist vertical candidates in this phase, which are strictly below the current row of e then their places of change are also strictly below the current row of e. See the proof of Claim 3.)

We denote the application of MERGE 3 to (T,H) restricted to the vertical pointers in V_1 by MERGE 3.1 and the application of MERGE 3 to (T,H) restricted to the vertical pointers in V_2 by MERGE 3.2. Then the application of MERGE 3 is equivalent to the simultaneous application of MERGE 3.1 and MERGE 3.2, where MERGE 3.2 starts after the vertical pointers that came from V_1 are strictly above of the k-th row.

By induction hypothesis we reobtain (T,H) if we apply SPLIT 3 simultaneously to the protagonists of MERGE 3.1 including the minimal horizontal candidates e' from the if-condition (we denote this restricted application of SPLIT 3 by SPLIT 3.1) and to the protagonists of MERGE 3.2 including these horizontal candidates (we denote this restricted application of SPLIT 3 by SPLIT 3.2) such that when a candidate in SPLIT 3.1 moves weakly below the k-th row the application of SPLIT 3.2 has already finished. Note that e is not touched from above in the application of SPLIT 3 to (T,H) (if we apply it to all candidates at once), for e does not touch from below in the application of MERGE 3 to (T,H). Consequently if we apply SPLIT 3 unrestricted to (T,H), then this application starts with the application of SPLIT 3.1, until we are in Case 3 of SPLIT 3 or SPLIT 3.1 has terminated. In the first case the e' in the description of Case 3 in SPLIT 3 equals e and we perform SPLIT 3.2. After that we finish the application SPLIT 3.1 in this case. In the second case the set C' is empty at the beginning of the step of SPLIT 3 immediately after the termination of SPLIT 3.1 and e is the candidate in C with minimal row. Thus we terminate with the application of SPLIT 3.2 in this case. $\square$

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