ENUMERATION OF RHOMBUS TILINGS OF A HEXAGON WHICH CONTAIN A FIXED RHOMBUS IN THE CENTRE

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Abstract. We compute the number of rhombus tilings of a hexagon with side lengths a,b,c,a,b,c which contain the central rhombus and the number of rhombus tilings of a hexagon with side lengths a,b,c,a,b,c which contain the 'almost central' rhombus above the centre.

1. Introduction

Let a, b and c be positive integers and consider a hexagon with side lengths a,b,c,a,b,c whose angles are 120° (see Figure 1).

The subject of our interest is rhombus tilings of such a hexagon using rhombi with all sides of length 1 and angles 60° and 120°. Figure 2 shows an example of a rhombus tiling of a hexagon with $a = 3$, $b = 5$ and $c = 4$.

A first natural question to be asked is how many rhombus tiling of a fixed hexagon exist. A well known bijection between such rhombus tilings and plane partitions contained in an $a \times b \times c$ box [3] and MacMahon's enumeration of plane partitions [11, Sec. 429, $q \to 1$; proof in Sec. 494] give the following answer: *The number of all rhombus tilings of a hexagon with side lengths a,b,c,a,b,c equals*

$$\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}. \tag{1.1}$$

On a next level, one may ask for the number of rhombus tilings with special properties. In this paper, we address this question. We study rhombus tilings of a hexagon which contain the central rhombus and rhombus tilings of a hexagon which contain the 'almost central' rhombus above the centre. By the 'central rhombus' we mean the rhombus whose centre is equal to the centre of the hexagon (see Figure 3, where the central rhombus is marked; furthermore, the example of a rhombus tiling in Figure 2 contains the central rhombus). By the 'almost central' rhombus above the centre we mean the horizontal rhombus whose lowest vertex is the centre of the hexagon (see Figure 4).

The main results of this paper are the following two theorems.

Theorem 1. *Let a,b,c be positive integers with $a \equiv b \pmod 2$ and $a \not\equiv c \pmod 2$. Then the number of rhombus tilings of a hexagon with side lengths a,b,c,a,b,c which contain the*

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rhombus in the centre is

$$\left(\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}\right)\frac{(1)_c}{(b+1)_{c+a-1}}\binom{\frac{b+c-1}{2}}{\frac{b-1}{2}}\binom{\frac{a+b+c-2}{2}}{\frac{b-1}{2}}2^{a-1}$$
$$\times\sum_{k=0}^{(a-1)/2}\left[\left(\frac{c+1}{2}\right)_k\left(\frac{1+b+c}{2}\right)_k\left(\frac{c+2k+2}{2}\right)_{(a-2k-1)/2}\right.$$
$$\left.\times\left(\frac{b+c+2k+3}{2}\right)_{(a-2k-1)/2}\frac{(\frac{1}{2})_{(a-2k-1)/2}}{(1)_{(a-2k-1)/2}}\right] \tag{1.2}$$

in case that a is odd, and

$$\left(\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}\right)\frac{b(1)_c}{(b+1)_{c+a-1}}\binom{\frac{b+c-1}{2}}{\frac{b}{2}}\binom{\frac{a+b+c-1}{2}}{\frac{b}{2}}2^{a-2}$$
$$\times\sum_{k=0}^{(a-2)/2}\left[\left(\frac{c+2}{2}\right)_k\left(\frac{1+b+c}{2}\right)_k\left(\frac{c+2k+3}{2}\right)_{(a-2k-2)/2}\right.$$
$$\left.\times\left(\frac{b+c+2k+3}{2}\right)_{(a-2k-2)/2}\frac{(\frac{1}{2})_{(a-2k-2)/2}}{(1)_{(a-2k-2)/2}}\right] \tag{1.3}$$

in case that a is even, where the Pochhammer symbol $(a)_k$ is defined by $(a)_k := a(a+1)\dots(a+k-1)$ if $k \geq 1$ and $(a)_0 := 1$.

The special case $a = b$ in Theorem 1 was previously derived in [1, Theorem 1, 2]. (In order to see that the two sums in Theorem 1 are equal to two sums in Theorem 1 and Theorem 2 from [1] in this special case, one has to apply Bailey's transformation (5.3) for balanced ${}_4F_3$–series.)

The assumption about the parity of the side lengths a,b,c, i.e., that not all side lengths of the hexagon have the same parity, comes from the fact that this condition is necessary for the existence of a central rhombus in a rhombus tiling of a hexagon with side lengths a,b,c,a,b,c. There is also a result similar to Theorem 1 for a hexagon whose side lengths have the same parity. In this case we choose for the fixed rhombus the horizontal rhombus whose lowest vertex is the centre of the hexagon. (We could have also chosen the horizontal rhombus whose uppermost vertex is the centre of the hexagon). The number of these rhombus tilings is given by formulas quite similar to those in Theorem 1, and, furthermore, the proofs of these formulas are analogous to the proofs of the formulas in Theorem 1. We obtain the following:

Theorem 2. *Let a,b,c be positive integers with $a \equiv b \equiv c \pmod 2$. Then the number of rhombus tilings of hexagon with side lengths a,b,c,a,b,c which contain the horizontal rhombus* *whose lowest vertex is the centre of the hexagon is equal to*

$$\left(\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}\right)\frac{(1)_c}{(b+1)_{c+a-1}}\binom{\frac{b+c-2}{2}}{\frac{b-1}{2}}\binom{\frac{a+b+c-1}{2}}{\frac{b-1}{2}}2^{a-1}$$
$$\times\Bigg(\left(\frac{c+1}{2}\right)_{(a-1)/2}\left(\frac{b+c+2}{2}\right)_{(a-1)/2}\frac{\left(\frac{1}{2}\right)_{(a-1)/2}}{(1)_{(a-1)/2}}$$
$$+\sum_{k=1}^{(a-1)/2}\left(\frac{c+2}{2}\right)_{k-1}\left(\frac{b+c}{2}\right)_{k}\left(\frac{c+2k+1}{2}\right)_{(a-2k+1)/2}$$
$$\times\left(\frac{b+c+2k+2}{2}\right)_{(a-2k-1)/2}\frac{\left(\frac{1}{2}\right)_{(a-2k-1)/2}}{(1)_{(a-2k-1)/2}}\Bigg) \tag{1.4}$$

in case that a is odd, and

$$\left(\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}\right)\frac{(1)_c}{(b)_{c+a}}\binom{\frac{b+c-2}{2}}{\frac{b-2}{2}}\binom{\frac{a+b+c-2}{2}}{\frac{b-2}{2}}2^{a}$$
$$\times\Bigg(\left(\frac{c+2}{2}\right)_{(a-2)/2}\left(\frac{b+c+2}{2}\right)_{a/2}\frac{\left(\frac{1}{2}\right)_{a/2}}{(1)_{(a-2)/2}}$$
$$+\sum_{k=1}^{a/2}\left(\frac{c+1}{2}\right)_{k}\left(\frac{b+c}{2}\right)_{k}\left(\frac{c+2k+2}{2}\right)_{(a-2k)/2}$$
$$\times\left(\frac{b+c+2k+2}{2}\right)_{(a-2k)/2}\frac{\left(\frac{1}{2}\right)_{(a-2k)/2}}{(1)_{(a-2k)/2}}\Bigg) \tag{1.5}$$

in case that a is even.

The special case $a = b$ in Theorem 2 was previously derived in [4, Theorem 1, 2]. (In order to see that the two sums in Theorem 2 are equal to the two sums in Theorem 1 and Theorem 2 from [4] for this special case, one first has to apply a certain contiguous relation on the balanced ${}_4F_3$–series in Theorem 2. In each case we obtain the sum of a balanced ${}_3F_2$–series, which can be evaluated by Saalschütz's summation formula (see [13, (2.3.1.3)]), and another balanced ${}_4F_3$–series. An application of Bailey's transformation to the latter ${}_4F_3$–series finally shows the equivalence of the result in Theorem 2 and the result in [4, Theorem 1, 2] for the special case.)

These enumeration results for rhombus tilings with a fixed rhombus are not only interesting because they add more results to the growing set of results on the enumeration of rhombus tilings with special properties. These enumeration also contribute to the interesting question of what a 'typical' rhombus tiling of a hexagon looks like. Cohen, Larsen und Propp address this question in [2]. We are able to deduce the following two theorems from Theorem 1 and Theorem 2. They confirm Conjecture 1 in [2] for a special case.

Theorem 3. *Let α, β and γ be non-negative real numbers. Then the probability that a rhombus tiling of the hexagon with side lengths $a, b, c, a, b, c,$ where $a \sim \alpha N$, $b \sim \beta N$, $c \sim \gamma N$*

and $a \equiv b \not\equiv c \pmod 2$, contains the rhombus in the centre is asymptotically

$$\frac{2}{\pi}\arcsin\left(\sqrt{\frac{\alpha\beta}{(\beta+\gamma)(\alpha+\gamma)}}\right) \tag{1.6}$$

as N tends to infinity.

Theorem 4. *Let α, β and γ be non-negative real numbers. Then the probability that a rhombus tiling of the hexagon with side lengths a,b,c,a,b,c, where $a \sim \alpha N$, $b \sim \beta N$, $c \sim \gamma N$ and $a \equiv b \equiv c \pmod 2$, contains the 'almost central' rhombus above the centre is asymptotically*

$$\frac{2}{\pi}\arcsin\left(\sqrt{\frac{\alpha\beta}{(\beta+\gamma)(\alpha+\gamma)}}\right) \tag{1.7}$$

as N tends to infinity.

Roughly speaking, Conjecture 1 in [2] predicts the following: Fix an arbitrary point (x,y) in the hexagon with side lengths a,b,c,a,b,c. Then the probability that a random rhombus tiling of a hexagon contains a vertical rhombus at this point tends to $\mathcal{P}_{\alpha,\beta,\gamma}(x,y)$ as the hexagon becomes large. Here, $\mathcal{P}_{\alpha,\beta,\gamma}$ is a certain function (defined in [2, Theorem 1]) that depends on the proportions α,β,γ of the side lengths of the hexagon (see Theorem 3). Theorem 3 and Theorem 4 confirm this conjecture for the centre of the hexagon. (The reader should know that Cohen, Larsen and Propp use another coordinate system in [2] than we do and therefore they are in the position to speak of vertical rhombi. But of course, their result can easily be translated into our coordinate system where we fix horizontal rhombi instead of vertical rhombi.)

We want to direct the reader's attention to a side result of the present work, given in Lemma 2. It expresses, for *any* fixed rhombus, the number of rhombus tilings containing that fixed rhombus as a triple sum. I am currently pursuing an asymptotic analysis of this triple sum with the ultimate goal of effectively proving Cohn, Larsen and Propp's conjecture.

In order to prove Theorem 1 and Theorem 2 we make use of a bijection between rhombus tilings which contain the central rhombus, respectively rhombus tilings which contain the 'almost central' rhombus above the centre, and non-intersecting lattice paths. This bijection actually works for arbitrary rhombi, i.e., the fixed rhombus need not be placed in the centre or next to the centre. The description of this bijection is the subject of Section 2. Since the number of non-intersecting lattice paths is given by a determinant due to Lindström, Gessel and Viennot (see Theorem 5), this bijection reduces the problem to evaluating a certain determinant with binomial entries, see Lemma 1. In case that the fixed rhombus is situated in the centre or next to the centre we are able to evaluate the determinant, see Lemma 3 and Lemma 4 in Section 4. We partly make us of a method that has already produced evaluations of other binomial determinants (see e.g. [4], [9]). In the course of evaluating the determinant we need an alternative expression for the number of rhombus tilings that we are interested in, in form of the aforementioned triple sum. This is given in Lemma 2 in Section 3. Finally, in Section 5 we provide the proofs of Theorem 3 and Theorem 4.

2. From rhombus tilings to non-intersecting lattice paths and determinants

As mentioned before, the subject of this section is the bijection between non-intersecting lattice paths and rhombus tilings. In our context, non-intersecting lattice paths are disjoint paths in the lattice $\mathbb{Z}^2$ with steps in direction $(1,0)$ and $(0,-1)$. Figure 6 shows an example of such a family of non-intersecting lattice paths. It consists of three paths, each one connecting an initial point A_i with a destination point E_i, $i = 1, 2, 3$.

Figure 5 and Figure 6 illustrate the bijection between rhombus tilings of a hexagon with side lengths a,b,c,a,b,c and non-intersecting lattice paths with initial points

$$A_i = (i - 1, c + i - 1) \tag{2.1}$$

and destination points

$$E_i = (b + i - 1, i - 1), \tag{2.2}$$

$i = 1, 2, \ldots, a$. Figure 5 shows the rhombus tiling from Figure 2 and an indication of the corresponding non-intersecting lattice paths. The initial points A_i' and the destination points E_i' of the paths in this figure are the centres of the sides of the rhombi that form the two sides of the hexagon with side lengths a. Roughly speaking these paths just describe 'the way down' from A_i' to E_i' on the three dimensional pile of cubes — elsewhere called *plane partitions* — which the rhombus tiling, when interpreted as three-dimensional object, gives.

In order to obtain the corresponding family of non-intersecting lattice paths in Figure 6, we only have to change the $120°$ angles of the paths in Figure 5 into right angles.

Therefore we already have a bijection between all rhombus tilings of a hexagon with side lengths a,b,c,a,b,c and all non-intersecting lattice paths starting in A_i and stopping in E_i. However, we are interested in special rhombus tilings — namely in those with one fixed rhombus. Because of that we have to look for a simple description of the non-intersecting lattice paths that correspond to the rhombus tilings with a fixed rhombus. Studying Figure 5 and Figure 6 again, we see that the marked rhombus in Figure 5 corresponds to the edge $(x - 1, y) \to (x, y)$ in Figure 6 (in our example $x = 4$ and $y = 3$). Therefore the rhombus tilings which contain a fixed horizontal rhombus correspond to the families of non-intersecting lattice paths which contain a fixed horizontal edge. Thus, we have found a bijection between rhombus tilings of a hexagon with side lengths a,b,c,a,b,c which contain a fixed horizontal rhombus and non-intersecting lattice paths with initial points A_i and destination points E_i, $i = 1, 2, \ldots, a$, which contain a fixed horizontal edge. As we will see in a moment, the latter can be counted with the help of the following main theorem on non-intersecting lattice paths:

Theorem 5 (Lindström-Gessel-Viennot)**.** *Let $A_1, A_2, \ldots, A_r$ and $E_1, E_2, \ldots, E_r$ be points in $\mathbb{Z}^2$. Then the determinant*

$$\det_{1 \le i,j \le r} \left(|P\,(A_i \to E_j)| \right) \tag{2.3}$$

is equal to

$$\sum_{\sigma \in \mathcal{S}_r} \operatorname{sgn} \sigma \cdot \left| P^+\,(A \to E_\sigma) \right|,$$

where $|P\,(A_i \to E_j)|$ denotes the number of lattice paths connecting A_i to E_j, and $|P^+\,(A \to E_\sigma)|$ denotes the number of families $(P_1, P_2, \ldots, P_r)$ of non-intersecting lattice paths, the ith path P_i connecting A_i to $E_{\sigma(i)}$, $1 \le i \le r$.

This theorem can be found in [6, Theorem 1] or, alternatively, in [10, Lemma 1].

Remark 1. Usually the theorem is applied to the following situation: There exists only one permutation σ (normally the identity permutation) such that the set of families of non-intersecting lattice paths connecting A_i to $E_{\sigma(i)}$, $1 \le i \le r$, is not empty (e.g., this is the case for the points A_i and E_i defined in (2.1) and (2.2)). Then Theorem 5 solves the enumeration problem of counting non-intersecting lattice paths — by means of the determinant (2.3) — totally. This is because $|P\,(A \to E)|$ is easy to determine: Let $A = (a_1, a_2)$ and $E = (e_1, e_2)$

be two points in $\mathbb{Z}^2$, such that A is located in the north-west of E (i.e., $e_1 \geq a_1$ and $a_2 \geq e_2$). Then the number of lattice paths with steps in direction $(1,0)$ and $(0,-1)$ is

$$|P(A \to E)| = \binom{e_1 - a_1 + a_2 - e_2}{e_1 - a_1} = \binom{e_1 - a_1 + a_2 - e_2}{a_2 - e_2}, \tag{2.4}$$

since each lattice paths corresponds to a choice of $e_1 - a_1$ steps in direction $(1,0)$ out of the total number of $e_1 - a_1 + a_2 - e_2$ steps.

Remark 1 shows that Theorem 5 can be useful in the enumeration of non-intersecting lattice paths with given initial and destination points. Since our enumeration problem of non-intersecting lattice paths not only involves fixed initial and destination points but also the additional condition about the fixed edge $(x-1,y) \to (x,y)$, Theorem 5 seems useless in our situation at first glance. The following 'trick' remedies this matter: We add the following pair of initial and destination points:

$$A_{a+1} = (x,y) \qquad \text{and} \qquad E_{a+1} = (x-1,y) \tag{2.5}$$

(see Figure 6). Then, as is not difficult to see, the number of rhombus tilings with a fixed rhombus 'at' (x,y) equals the number of families of non-intersecting lattice paths with initial points A_i and destination points E_i, $i = 1,2,\ldots,a+1$.

Now we are ready to apply Theorem 5 to the points A_i and E_i, $i = 1,2,\ldots,a+1$, defined in (2.1), (2.2) and (2.5). In order to figure out what the determinant (2.3) actually gives, we have to find the permutations σ for which there exist non-intersecting lattice paths connecting A_i to $E_{\sigma(i)}$ with $1 \leq i \leq a+1$. It is quite easy to see that this is only accomplished by transpositions of the form $(i, a+1)$, where $i \neq a+1$. The fact that all transpositions have the same sign — namely -1 — implies that the determinant in (2.3) applied to our special points gives the number of families of non-intersecting lattice paths with initial points A_i and destination points E_i, $1 \leq i \leq a+1$, with negative sign.

We use (2.4) to compute $|P(A_i \to E_j)|$ in (2.3), and finally obtain the following:

Lemma 1. *Let a,b and c be positive integers and (x,y) be an integer point such that $0 \leq x \leq b+a-1$ and $1 \leq y \leq c+a-1$. Then the number of rhombus tilings of a hexagon with side lengths a,b,c,a,b,c which contain the fixed horizontal rhombus that corresponds to the point (x,y) in the bijection described above equals*

$$-\det_{1 \leq i,j \leq a+1} \begin{array}{cc} \begin{array}{cc} 1 \leq j \leq a & j = a+1 \end{array} & \\ \left(\begin{array}{c|c} \binom{b+c}{c-i+j} & \binom{b-x+y}{y-i+1} \\ \hline \binom{c+x-y-1}{x-j} & 0 \end{array} \right) & \begin{array}{l} 1 \leq i \leq a \\ i = a+1 \end{array} \end{array}. \tag{2.6}$$

In Lemma 1 we refer to a correspondence between the integer points (x,y), $0 \leq x \leq b+a-1$ and $1 \leq y \leq c+a-1$, and the horizontal rhombi we use for the rhombus tilings of a hexagon with side lengths a,b,c,a,b,c. This correspondence is implicitly given by the bijection between rhombus tilings and non-intersecting lattice paths described above. The following remark makes the bijection between points and rhombi more explicit.

Remark 2. Again we consider a hexagon with side lengths a,b,c,a,b,c. We introduce the following oblique angled coordinate system: Its origin is located in one of the two vertices, where sides of lengths b and c meet, and the axes are induced by those two sides (see Figure 7).

The units are chosen such that the (Euclidean) side lengths of the considered hexagon are a,b,c,a,b,c in this coordinate system, too. (That is to say, the two triangles in Figure 7 with vertices in the origin form the unit 'square'). Thus, in this coordinate system, the points in Figure 5 have coordinates $A'_1 = (1/2, 9/2)$, $A'_2 = (3/2, 11/2)$, $A'_3 = (5/2, 13/2)$, $E'_1 = (11/2, 1/2)$, $E'_2 = (13/2, 3/2)$ and $E'_3 = (15/2, 5/2)$. Phrased differently, the coordinate system was chosen such that $A'_i = A_i + (1/2, 1/2)$ and $E'_i = E_i + (1/2, 1/2)$, where A'_i and E'_i are the initial and destination points of the paths in Figure 5, and A_i and E_i are defined as in (2.1), (2.2) and (2.5), $1 \le i \le a+1$. From this point of view, the family of paths in Figure 5 is the family of paths in Figure 6 translated by $(1/2, 1/2)$ and drawn in the oblique angled coordinate system. Furthermore, we see that the integer point (x, y) in the oblique angled coordinate system is just the lowest vertex of the corresponding rhombus under the aforementioned bijection between points and rhombi (see Figure 5 and Figure 6).

3. From the determinant to a triple sum

The aim of this section is the derivation of a triple sum that is equal to (2.6) and therefore gives the number of all rhombus tilings of a hexagon with side lengths a,b,c,a,b,c which contain a fixed rhombus with lowest vertex (x, y):

Lemma 2. *Let a, b and c be positive integers, and let (x, y) be an integer point such that $0 \le x \le b+a-1$ and $1 \le y \le c+a-1$. Then the number of rhombus tilings of a hexagon with side lengths a,b,c,a,b,c which contain the fixed horizontal rhombus with lowest vertex (x, y) (in the oblique angled coordinate system; see Remark 2) equals*

$$\left(\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}\right)\frac{(1)_c}{(b+1)_c}$$
$$\times \sum_{n=1}^{a}\sum_{m=1}^{a}\sum_{s=1}^{m}\left[(-1)^{n+s}\binom{c+x-y+n-2}{x-1}\binom{b-x+y+s-1}{b-x+s-1}\frac{(b+1)_{s-1}}{(b+c+1)_{s-1}}\right.$$
$$\left.\binom{m-1}{s-1}\frac{(c+1)_{n-1}}{(n-1)!}\frac{(b+c+n)_{m-n}}{(m-n)!}\right]. \tag{3.1}$$

Outline of the proof of Lemma 2. We derive the triple sum by starting from (2.6). The matrix underlying the determinant in (2.6) has a 'homogeneous' definition, except for the last row and the last column. Our proof starts with some elementary row and column operations that transform the 'homogeneous' submatrix into a matrix of triangular form (see (3.10)). Next we expand the determinant along the exceptional row and along the exceptional column (see (3.11)). The result is a triple sum with a summand that involves another determinant. But this determinant is the determinant of the aforementioned triangular matrix with one row and one column deleted (which row and which column depends on the summation index of the triple sum, see (3.13)) and therefore we are able to compute it: The triangle property allows an expansion of this remaining determinant along the first $(n-1)$ columns and the last $(a-m)$ rows, where m denotes the missing row and n denotes the missing column (see (3.14)). Finally the remaining $(m-n) \times (m-n)$ determinants can be reduced to Vandermonde's determinant (see (3.15)).

Proof of Lemma 2 – The details. As mentioned before, we first describe some elementary row and column operations that transform the 'homogeneous' $a \times a$ submatrix $\left(\binom{b+c}{c-i+j}\right)_{1 \le i,j \le a}$ of the matrix underlying the determinant in (2.6) into an upper triangular matrix.

We begin with some elementary column operations: By using the symmetry of the binomial coefficient (i.e., the identity $\binom{n}{k}=\binom{n}{n-k}$) in the last row of the matrix underlying the determinant in (2.6) we observe that the determinant in (2.6) is equal to

$$-\det_{1\le i,j\le a+1}\begin{array}{cc} \begin{array}{cc} \quad 1\le j\le a \quad & \quad j=a+1 \quad \end{array} & \\ \left(\begin{array}{c|c} \binom{b+c}{c-i+j} & \binom{b-x+y}{y-i+1} \\ \hline \binom{c+x-y-1}{c-y+j-1} & 0 \end{array}\right) & \begin{array}{l} 1\le i\le a \\ i=a+1 \end{array} \end{array}. \tag{3.2}$$

We add the $(j-1)$th column to the jth column, $j=a,a-1,\dots,2$, in that order. The entries of the changed matrix read as

$$\binom{b+c}{c-i+j}+\binom{b+c}{c-i+j-1}=\binom{b+c+1}{c-i+j}$$

for $i=1,2,\dots,a$ and $j=2,3,\dots,a$, and

$$\binom{c+x-y-1}{c-y+j-1}+\binom{c+x-y-1}{c-y+j-2}=\binom{c+x-y}{c-y+j-1}$$

for $i=a+1$ and $j=2,3,\dots,a$. The other entries do not change. Thus, the following determinant is equal to the determinant in (3.2):

$$-\det_{1\le i,j\le a+1}\begin{array}{cc} \begin{array}{ccc} \quad j=1 \quad & \quad 2\le j\le a \quad & \quad j=a+1 \quad \end{array} & \\ \left(\begin{array}{c|c|c} \binom{b+c}{c-i+1} & \binom{b+c+1}{c-i+j} & \binom{b-x+y}{y-i+1} \\ \hline \binom{c+x-y-1}{c-y} & \binom{c+x-y}{c-y+j-1} & 0 \end{array}\right) & \begin{array}{l} 1\le i\le a \\ i=a+1 \end{array} \end{array}. \tag{3.3}$$

Next we repeat the procedure, i.e., we add the $(j-1)$th column to the jth column, $j=a,a-1,\dots,3$, in that order. Thus, we obtain that the following determinant is equal to the determinant in (3.3):

$$-\det_{1\le i,j\le a+1}\begin{array}{cc} \begin{array}{cccc} \quad j=1 \quad & \quad j=2 \quad & \quad 3\le j\le a \quad & \quad j=a+1 \quad \end{array} & \\ \left(\begin{array}{c|c|c|c} \binom{b+c}{c-i+1} & \binom{b+c+1}{c-i+2} & \binom{b+c+2}{c-i+j} & \binom{b-x+y}{y-i+1} \\ \hline \binom{c+x-y-1}{c-y} & \binom{c+x-y}{c-y+1} & \binom{c+x-y+1}{c-y+j-1} & 0 \end{array}\right) & \begin{array}{l} 1\le i\le a \\ i=a+1 \end{array} \end{array}.$$

We repeat this procedure of adding successive columns from right to left each time stopping one column earlier than before, as long as possible. This means that the procedure is performed $a-1$ times including the two steps described in detail. Since the upper parameter of the binomial entry increases by one every time it is involved and every entry in the jth column participates $j-1$ times exactly, $1\le j\le a$, this procedure yields the following determinant:

$$-\det_{1\le i,j\le a+1}\begin{array}{c c}\begin{array}{c|c} 1\le j\le a & j=a+1 \end{array} & \\ \left(\begin{array}{c|c} \binom{b+c+j-1}{c-i+j} & \binom{b-x+y}{y-i+1} \\ \hline \binom{c+x-y+j-2}{c-y+j-1} & 0 \end{array}\right) & \begin{array}{l} 1\le i\le a \\ i=a+1 \end{array}\end{array}. \tag{3.4}$$

Now we apply some elementary row operations to (3.4). In fact these row operations are analogous to the column operations we just applied to (3.2). In order to do so, we first use the symmetry of the binomial coefficient for the first a rows of the determinant in (3.4) and observe that the determinant

$$-\det_{1\le i,j\le a+1}\begin{array}{c c}\begin{array}{c|c} 1\le j\le a & j=a+1 \end{array} & \\ \left(\begin{array}{c|c} \binom{b+c+j-1}{b+i-1} & \binom{b-x+y}{b+i-x-1} \\ \hline \binom{c+x-y+j-2}{c-y+j-1} & 0 \end{array}\right) & \begin{array}{l} 1\le i\le a \\ i=a+1 \end{array}\end{array} \tag{3.5}$$

is equal to the determinant in (3.4). As announced, we now add the $(i-1)$th row of the determinant in (3.5) to the ith row, starting at $i=a$ and stopping at $i=2$. Next we do the same with the resulting determinant, starting at $i=a$ but stopping at $i=3$. After repeating this procedure $a-1$ times we obtain that the following determinant is equal to the determinant in (3.5):

$$-\det_{1\le i,j\le a+1}\begin{array}{c c}\begin{array}{c|c} 1\le j\le a & j=a+1 \end{array} & \\ \left(\begin{array}{c|c} \binom{b+c+i+j-2}{b+i-1} & \binom{b-x+y+i-1}{b+i-x-1} \\ \hline \binom{c+x-y+j-2}{c-y+j-1} & 0 \end{array}\right) & \begin{array}{l} 1\le i\le a \\ i=a+1 \end{array}\end{array}. \tag{3.6}$$

Now we take the factor $(-1)^{b+c+i-1}(b+c+i-1)!/(b+i-1)!$ out of the ith row of the determinant in (3.6), $i=1,2,\dots,a$. This yields

$$-\prod_{i=1}^{a}(-1)^{b+c+i-1}\frac{(b+c+i-1)!}{(b+i-1)!} \times \det_{1\le i,j\le a+1}\begin{array}{c c}\begin{array}{c|c} 1\le j\le a & j=a+1 \end{array} & \\ \left(\begin{array}{c|c} \frac{(j-1)!}{(c+j-1)!}\binom{-j}{b+c+i-1} & (-1)^{b+c+i-1}\frac{(b+i-1)!}{(b+c+i-1)!}\binom{b-x+y+i-1}{b+i-x-1} \\ \hline \binom{c+x-y+j-2}{c-y+j-1} & 0 \end{array}\right) & \begin{array}{l} 1\le i\le a \\ i=a+1 \end{array}\end{array}. \tag{3.7}$$

Finally we want to apply the elementary row operations we just applied to the determinant in (3.5) once again. An analysis of the elementary row operations that we applied to the determinant in (3.5) yields that the determinant in (3.6) was produced from the determinant in (3.5) by replacing the ith row by

$$\sum_{s=1}^{i} \binom{i-1}{s-1} A(s),$$

$1 \le i \le a$, where $A(s)$ denotes the sth row of the determinant in (3.5). We perform this replacement of the entries in the determinant in (3.7) and obtain

$$- \prod_{i=1}^{a} (-1)^{b+c+i-1} \frac{(b+c+i-1)!}{(b+i-1)!} \times \det_{1 \le i,j \le a+1} \begin{array}{c} \\ \left(\begin{array}{c|c} \frac{(j-1)!}{(c+j-1)!} \binom{i-j-1}{b+c+i-1} & \sum_{s=1}^{i} \binom{i-1}{s-1} (-1)^{b+c+s-1} \frac{(b+s-1)!}{(b+c+s-1)!} \binom{b-x+y+s-1}{b+s-x-1} \\ \hline \binom{c+x-y+j-2}{c-y+j-1} & 0 \end{array} \right) \\ \scriptstyle 1 \le j \le a \qquad\qquad\qquad j = a+1 \end{array} \begin{array}{l} \scriptstyle 1 \le i \le a \\ . \\ \scriptstyle i = a+1 \end{array} \tag{3.8}$$

The entry in the ith row and jth column, $1 \le i \le a$ and $1 \le j \le a$, develops from the corresponding entry in (3.7) by using Vandermonde's summation formula:

$$\sum_{s=1}^{i} \binom{i-1}{s-1} \frac{(j-1)!}{(c+j-1)!} \binom{-j}{b+c+s-1} = \frac{(j-1)!}{(c+j-1)!} \binom{i-j-1}{b+c+i-1}.$$

Next we apply the elementary identity

$$\binom{n}{k} = \binom{-n+k-1}{k} (-1)^k \tag{3.9}$$

to this entry in the ith row and jth column of the matrix underlying the determinant in (3.8), $1 \le i \le a$ and $1 \le j \le a$, and then use the symmetry of the binomial coefficient. Finally we take the factor $(-1)^{b+c+i-1}$ out of the ith row, $1 \le i \le a$, and obtain that the expression in (3.8) is equal to

$$- \prod_{i=1}^{a} \frac{(b+c+i-1)!}{(b+i-1)!} \times \det_{1 \le i,j \le a+1} \begin{array}{c} \scriptstyle 1 \le j \le a \qquad\qquad\qquad j = a+1 \\ \left(\begin{array}{c|c} \frac{(j-1)!}{(c+j-1)!} \binom{b+c+j-1}{j-i} & \sum_{s=1}^{i} \binom{i-1}{s-1} (-1)^{i-s} \frac{(b+s-1)!}{(b+c+s-1)!} \binom{b-x+y+s-1}{b+s-x-1} \\ \hline \binom{c+x-y+j-2}{c-y+j-1} & 0 \end{array} \right) \end{array} \begin{array}{l} \scriptstyle 1 \le i \le a \\ . \\ \scriptstyle i = a+1 \end{array} \tag{3.10}$$

With pleasure we discover that the $a \times a$ submatrix induced by the first a rows and first a columns of the matrix underlying the determinant in (3.10) is an upper triangular matrix. This is because the binomial coefficient $\binom{\alpha}{k}$ is defined to be zero if $k < 0$ for any indeterminante α. Therefore the determinant is of that form we were looking for at the beginning of the proof.

Next we expand (3.10) along the last row and then along the last column, to obtain

$$-\prod_{i=1}^{a}\frac{(b+c+i-1)!}{(b+i-1)!}\sum_{n=1}^{a}(-1)^{a+1+n}\binom{c+x-y+n-2}{c-y+n-1}$$
$$\times\sum_{m=1}^{a}(-1)^{a+m}\sum_{s=1}^{m}\binom{m-1}{s-1}(-1)^{m-s}\frac{(b+s-1)!}{(b+c+s-1)!}\binom{b-x+y+s-1}{b+s-x-1}$$
$$\times\det_{\substack{1\le i,j\le a\\ i\ne m,j\ne n}}\left(\frac{(j-1)!}{(c+j-1)!}\binom{b+c+j-1}{j-i}\right). \tag{3.11}$$

Then we take the factor $(j-1)!/(c+j-1)!$ out of the jth column of the remaining determinant. This, together with some other manipulations, gives

$$\prod_{i=1}^{a}\frac{(b+c+i-1)!(i-1)!}{(b+i-1)!(c+i-1)!}$$
$$\times\sum_{n=1}^{a}\sum_{m=1}^{a}\sum_{s=1}^{m}(-1)^{n+s}\binom{c+x-y+n-2}{x-1}\binom{b-x+y+s-1}{b+s-x-1}\binom{m-1}{s-1}$$
$$\times\frac{(b+s-1)!}{(b+c+s-1)!}\frac{(c+n-1)!}{(n-1)!}\det_{\substack{1\le i,j\le a\\ i\ne m,j\ne n}}\left(\binom{b+c+j-1}{j-i}\right). \tag{3.12}$$

Now we have to compute

$$\det_{\substack{1\le i,j\le a\\ i\ne m,j\ne n}}\left(\binom{b+c+j-1}{j-i}\right). \tag{3.13}$$

This is a determinant of an upper triangular matrix, where the mth row and nth column was deleted. Therefore it is only different from zero if $n\le m$. So, let us assume $n\le m$. Expansion of the determinant along the first $(n-1)$ columns and along the last $(a-m)$ rows yields

$$\det_{\substack{n+1\le j\le m\\ n\le i\le m-1}}\left(\binom{b+c+j-1}{j-i}\right)=\det_{\substack{n\le j\le m-1\\ n\le i\le m-1}}\left(\binom{b+c+j}{j-i+1}\right) \tag{3.14}$$

for the determinant in (3.13).

We are going to reduce this determinant to Vandermonde's determinant. In order to do so, we take $(b+c+j)!/(j-n+1)!$ out of the jth column, $1\le j\le a$, and $1/(b+c+i-1)!$ out of the ith row, $1\le i\le a$. Thus, we obtain that the determinant in (3.14) is equal to

$$(b+c+n)_{m-n}\prod_{j=1}^{m-n}\frac{1}{j!}\det_{\substack{1\le j\le m-n\\ 1\le i\le m-n}}\left((j-i+2)_{i-1}\right). \tag{3.15}$$

The entries of this determinant are monic polynomials in j of degree $i-1$, where i and j denote as usual the index of the row and the column of the entry. It is now straightforward to reduce this determinant by appropriate row operations to Vandermonde's determinant,

$$\det_{\substack{1\le i\le m-n\\ 1\le j\le m-n}}\left((j-i+2)_{i-1}\right)=\det_{\substack{1\le i\le m-n\\ 1\le j\le m-n}}\left(j^{i-1}\right)=\prod_{i=1}^{m-n-1}i!. \tag{3.16}$$

The last equation was obtained by applying the well-known formula for Vandermonde's determinant to $\det_{1\le i\le m-n, 1\le j\le m-n}\left(j^{i-1}\right)$.

Using (3.16) in (3.15), we obtain that the determinant in (3.13) equals

$$(b+c+n)_{m-n}\prod_{j=1}^{m-n}\frac{1}{j!}\prod_{i=1}^{m-n-1} i! = \frac{(b+c+n)_{m-n}}{(m-n)!}.$$

Accordingly, we replace the determinant in (3.12) by its value $(b+c+n)_{m-n}/(m-n)!$. Thus, we obtain the following triple sum for the determinant in (2.6):

$$\left(\prod_{i=2}^{a}\frac{(b+c+i-1)!(i-1)!}{(b+i-1)!(c+i-1)!}\right) \\ \times\sum_{n=1}^{a}\sum_{m=1}^{a}\sum_{s=1}^{m}\left[(-1)^{n+s}\binom{c+x-y+n-2}{x-1}\binom{b-x+y+s-1}{b-x+s-1}\binom{m-1}{s-1}\right. \\ \left.\times\frac{(b+1)_{s-1}}{(b+c+1)_{s-1}}\frac{(c+1)_{n-1}}{(n-1)!}\frac{(b+c+n)_{m-n}}{(m-n)!}\right].$$

Therefore Lemma 2 is finally proved since

$$\prod_{i=2}^{a}\frac{(b+c+i-1)!(i-1)!}{(b+i-1)!(c+i-1)!} = \left(\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}\right)\frac{(1)_c}{(b+1)_c}.$$

4. Evaluation of the determinants

In this section we compute the determinant from Lemma 1 in case that the fixed rhombus is placed in the centre (see Lemma 3) and in case that the lowest vertex of the fixed rhombus is placed in the centre (see Lemma 4). The combination of these two lemmas and Lemma 1 then establish Theorem 1 and Theorem 2.

In order to do so, we first have to figure out which integer point (x, y) corresponds to the central rhombus, respectively the 'almost central' rhombus above the centre, via the bijection described in Section 2. In the oblique angled coordinate system introduced in Remark 2 the point $((a+b)/2, (a+c)/2)$ is the centre of the considered hexagon. Because of that, $((a+b)/2, (a+c-1)/2)$ is the lowest vertex of the central rhombus and therefore the integer point that corresponds to the central rhombus (see Remark 2). Accordingly, $((a+b)/2, (a+c)/2)$ is the lowest vertex of the 'almost central' rhombus above the centre.

The evaluations of both determinants, namely the determinant corresponding to the case that the fixed rhombus is placed in the centre, and the determinant corresponding to the case that the fixed rhombus is placed next to the centre on the other hand, are quite similar. Therefore we concentrate on the first case, see the following lemma. Lemma 4 at the end of this section is devoted to the second case, but there I only explain the major differences to the first case.

Lemma 3. *Let a,b,c be integers. Then the determinant*

$$
-\det_{1\le i,j\le a+1}\;
\begin{array}{c}
\begin{array}{cc} 1\le j\le a & j=a+1 \end{array}\\
\left(\begin{array}{c|c}
\dbinom{b+c}{c-i+j} & \dbinom{\frac{b+c-1}{2}}{\frac{c}{2}-i+\frac{a+1}{2}}\\
\hline
\dbinom{\frac{b+c-1}{2}}{\frac{c}{2}-\frac{a+1}{2}+j} & 0
\end{array}\right)
\end{array}
\begin{array}{l} 1\le i\le a\\ \\ i=a+1 \end{array}
$$

is equal to

$$
\begin{aligned}
&\left(\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}\right)\frac{(1)_c}{(b)_c}\frac{1}{(b+c+1)_{a-1}}\binom{\frac{b+c-1}{2}}{\frac{b-1}{2}}\binom{\frac{a+b+c-2}{2}}{\frac{b-1}{2}}2^{a-1}\\
&\times\sum_{k=0}^{(a-1)/2}\left[\left(\frac{c+1}{2}\right)_k\left(\frac{1+b+c}{2}\right)_k\left(\frac{c+2k+2}{2}\right)_{(a-2k-1)/2}\right.\\
&\qquad\qquad\left.\times\left(\frac{b+c+2k+3}{2}\right)_{(a-2k-1)/2}\frac{(\frac{1}{2})_{(a-2k-1)/2}}{(1)_{(a-2k-1)/2}}\right]
\end{aligned}
$$

in case that a is odd, and

$$
\begin{aligned}
&\left(\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}\right)\frac{(1)_c}{(b)_c}\frac{b}{(b+c+1)_{a-1}}\binom{\frac{b+c-1}{2}}{\frac{b}{2}}\binom{\frac{a+b+c-1}{2}}{\frac{b}{2}}2^{a-2}\\
&\times\sum_{k=0}^{(a-2)/2}\left[\left(\frac{c+2}{2}\right)_k\left(\frac{1+b+c}{2}\right)_k\left(\frac{c+2k+3}{2}\right)_{(a-2k-2)/2}\right.\\
&\qquad\qquad\left.\times\left(\frac{b+c+2k+3}{2}\right)_{(a-2k-2)/2}\frac{(\frac{1}{2})_{(a-2k-2)/2}}{(1)_{(a-2k-2)/2}}\right]
\end{aligned}
$$

in case that a is even.

Outline of the proof of Lemma 3. The following procedure is used for computing the determinant: The determinant depends on the side lengths a, b and c of the hexagon. More precisely: The dimension of the matrix underlying the determinant is $a+1$, and the entries are certain binomial coefficients that depend on b, c and a. First, in Step 1, we reduce the problem to the computation of a determinant with entries that are polynomials in b and c if we fix a, see (4.6), and therefore a determinant that is itself a polynomial in those two variables. The comparison of the determinant in Lemma 3 and the determinant in (4.6), the latter will be denoted by $\det(D_a(b,c))$, yields that we have to show that the determinant $\det(D_a(b,c))$ is a product of certain linear factors and an irreducible polynomial (over $\mathbb{Z}$) in b and c, see (4.7), respectively (4.8).

In Step 2 of our proof we show that every linear factor on the right-hand side of (4.7), respectively (4.8), is indeed a linear factor of $\det(D_a(b,c))$. I explain the used procedure by an example: We have to show, e.g., that the linear factor $(c+1)$ is a factor of $\det(D_a(b,c))$ in case that a is even (see (4.8)). It is a fundamental algebraic fact that, in order to show that, it suffices to show that $\det(D_a(b,-1)) = 0$ if a is even. (Here we use the fact that $\det(D_a(b,c))$ is a polynomial in c if we fix a). Clearly, a determinant of a matrix is equal to zero if and only if there exists a linear combination of rows, or, equivalently, of columns. Therefore we find a linear combination of rows of $D_a(b,-1)$ which vanishes, and then prove it, see (4.13). Proving in this case means to establish hypergeometric identities [1].

At this point it is worth mentioning that the procedure described so far gives a complete solution for determinants that factorise completely into linear factors.

In Step 3 of the proof we finally compute the irreducible polynomial, which will be denoted by $P_a(b,c)$. In order to do so, we first look for special values of b, where $P_a(b,c)$ is 'nice'. Indeed, we discovered that $P_a(b,c)$ factors completely into linear factors for $b = -c-k$, $k = 1, 3, \dots, 2\lfloor (a-1)/2 \rfloor + 1$. We work out these evaluations of $P_a(b,c)$ in (4.30) and (4.31), and subsequently prove them by making use of the triple sum derived in Section 3. As it turns out, the degree of $P_a(b,c)$ as a polynomial in b is exactly $\lfloor (a-1)/2 \rfloor$. Thus, the above $\lfloor (a-1)/2 \rfloor + 1$ evaluations suffice to compute $P_a(b,c)$ by using Lagrange interpolation, see (4.11) and (4.12).

Proof of Lemma 3 – The details.

Step 1: From our determinant to a determinant with polynomial entries.

In Section 2 it was finally shown that the number of rhombus tilings which contain a fixed horizontal rhombus with lowest vertex (x,y) is given by the following $(a+1)\times(a+1)$ determinant:

$$-\det_{1\le i,j\le a+1}\begin{array}{c}\begin{array}{cc} \scriptstyle 1\le j\le a & \scriptstyle j=a+1 \end{array}\\ \left(\begin{array}{c|c} \binom{b+c}{c-i+j} & \binom{b-x+y}{y-i+1} \\ \hline \binom{c+x-y-1}{x-j} & 0 \end{array}\right)\end{array}\begin{array}{l} \scriptstyle 1\le i\le a. \\ \scriptstyle i=a+1 \end{array} \tag{4.1}$$

With displeasure we observe that the matrix underlying this determinant has an exceptional row and an exceptional column, namely the $(a+1)$th in both cases. In the following I describe some elementary row and column operations which lead to a matrix having the same determinant but a 'homogeneous' definition:

1. First we want the entries in the $(a+1)$th row to be zero — except for the first entry. Since $\binom{c+x-y-1}{x-j}$ is the jth entry in the $(a+1)$th row, we have to subtract $\binom{c+x-y-1}{x-j}/\binom{c+x-y-1}{x-j+1} = (x-j+1)/(c-y+j-1)$ times the $(j-1)$th column from the jth, starting at $j=a$ and stopping at $j=2$. We obtain a new matrix with the desired behaviour in the $(a+1)$th row. The (i,j) entry of this new matrix is

$$\binom{b+c}{c-i+j} - \frac{x-j+1}{c-y+j-1}\binom{b+c}{c-i+j-1},$$

$i,j = 2,3,\dots,a$, it is $\binom{c+x-y-1}{x-1}$ for $i=a+1$ and $j=1$, and it is 0 for $i=a+1$ and $j\ne 1$. The other entries do not change.

2. Next we want to do the same for the $(a+1)$th column: Accordingly, since $\binom{b-x+y}{y-i+1}$ is the ith entry in the $(a+1)$th column, we subtract $\binom{b-x+y}{y-i+1}/\binom{b-x+y}{y-i+2} = (y-i+2)/(b+i-x-1)$ times the $(j-1)$th row from the jth row, again starting at the bottom — that is to say,

[1] At this point it is worth mentioning that I used C. Krattenthaler's *Mathematica* package HYP [8] to handle most of the hypergeometric identities within this paper.

$j=a,a-1,\dots,2$. Therefore the (i,j) entry of our new matrix is given as follows:

$$\begin{aligned}&\binom{b+c}{c-i+j}-\frac{x-j+1}{c-y+j-1}\binom{b+c}{c-i+j-1}\\ -&\frac{y-i+2}{b+i-x-1}\left(\binom{b+c}{c-(i-1)+j}-\frac{x-j+1}{c-y+j-1}\binom{b+c}{c-(i-1)+j-1}\right)\end{aligned}\tag{4.2}$$

if $i,j=2,3\dots,a$, it is $\binom{b-x+y}{y}$ for $j=a+1$ and $i=1$, and it is 0 for $j=a+1$ and $i\neq 1$. Again, the other entries do not change.

3. In summary, we obtain the following determinant,

$$\det_{1\le i,j\le a+1}\begin{pmatrix} * & * & * & * & \binom{b-x+y}{y}\\ * & & & & 0\\ * & & a_{ij} & & \vdots\\ * & & & & \vdots\\ \binom{c+x-y-1}{x-1} & 0 & \dots & \dots & 0\end{pmatrix},$$

where $*$ describes an entry which is of no interest for us, and where a_{ij} denotes the expression in (4.2). This form suggests an expansion of the determinant along the last row and along the last column. Therefore (4.1) is equal to

$$-\binom{c+x-y-1}{x-1}\binom{b-x+y}{y}\det_{2\le i,j\le a}(a_{ij}).\tag{4.3}$$

We have reduced our problem to computing $\det_{2\le i,j\le a}(a_{ij})$ and therefore the job to compute the determinant of a 'homogeneous' matrix, even if the entries are now more complex. The advantage is that we can now take several factors out of the determinant so that the remaining entries are polynomials: A simple calculation shows that

$$a_{ij}=\frac{(b+c)!(b+i-j+2)_{j-2}(c-i+j+2)_{a-j}}{(b+i-1)!(c-j+a+1)!(c-y+j-1)(b+i-x-1)}\times H,$$

where H is the polynomial

$$\begin{aligned}H=&(b+i-j+1)(c-i+j+1)(c-y+j-1)(b+i-x-1)\\ &-(c-i+j)(c-i+j+1)(x-j+1)(b+i-x-1)\\ &-(b+i-j)(b+i-j+1)(y-i+2)(c-y+j-1)\\ &+(b+i-j+1)(c-i+j+1)(x-j+1)(y-i+2).\end{aligned}\tag{4.4}$$

We take $(b+c)!/((b+i-1)!(b+i-x-1))$ out of the ith row, and $1/((c-j+a-1)!(c-y+j-1))$ out of the jth column of the matrix $(a_{ij})_{2\le i,j\le a}$. This gives

$$\begin{aligned}\det_{2\le i,j\le a}(a_{ij})=&\left(\prod_{i=2}^{a}\frac{(b+c)!}{(b+i-1)!(c-i+a+1)!(c-y+i-1)(b+i-x-1)}\right)\\ &\times\det_{2\le i,j\le a}\left((b+i-j+2)_{j-2}(c-i+j+2)_{a-j}\cdot H\right).\end{aligned}\tag{4.5}$$

Again we have reduced our problem to the evaluation of another 'homogeneous' determinant, namely

$$\det_{2\le i,j\le a}\left((b+i-j+2)_{j-2}(c-i+j+2)_{a-j}\cdot H\right).\tag{4.6}$$

But this one has the pleasing property to be a polynomial in b and c if we fix a (since the entries of the underlying matrix are). In the following, $D_a(b,c)$ denotes the matrix underlying the determinant in (4.6) evaluated at $x=(a+b)/2$ and $y=(a+c-1)/2$.

When I computed this determinant for small values of a, I was led to the following conjecture [2]: There holds

$$\begin{aligned}\det_{2\le i,j\le a}\left(D_a(b,c)\right) &= \left(\prod_{i=2}^{a-1}(1+b+c)_{i-1}\right)\left(\prod_{i=2}^{a}(i-1)!\right)2^{a-1}\\ &\quad \left(\frac{b+1}{2}\right)^2_{(a-1)/2}\left(\frac{c+2}{2}\right)_{(a-1)/2}\left(\frac{1+b+c}{2}\right)_{(a-1)/2}\\ &\quad\times\sum_{k=0}^{(a-1)/2}\left[\left(\frac{c+1}{2}\right)_k\left(\frac{1+b+c}{2}\right)_k\left(\frac{c+2k+2}{2}\right)_{(a-2k-1)/2}\right.\\ &\qquad\qquad\left.\times\left(\frac{b+c+2k+3}{2}\right)_{(a-2k-1)/2}\frac{(\frac{1}{2})_{(a-2k-1)/2}}{(1)_{(a-2k-1)/2}}\right]\end{aligned}\tag{4.7}$$

if a is odd, and

$$\begin{aligned}\det_{2\le i,j\le a}\left(D_a(b,c)\right) &= \left(\prod_{i=2}^{a-1}(1+b+c)_{i-1}\right)\left(\prod_{i=2}^{a}(i-1)!\right)2^{a-2}\\ &\quad b\left(\frac{b+2}{2}\right)^2_{(a-2)/2}\left(\frac{c+1}{2}\right)_{a/2}\left(\frac{1+b+c}{2}\right)_{a/2}\\ &\quad\times\sum_{k=0}^{(a-2)/2}\left[\left(\frac{c+2}{2}\right)_k\left(\frac{1+b+c}{2}\right)_k\left(\frac{c+2k+3}{2}\right)_{(a-2k-2)/2}\right.\\ &\qquad\qquad\left.\times\left(\frac{b+c+2k+3}{2}\right)_{(a-2k-2)/2}\frac{(\frac{1}{2})_{(a-2k-2)/2}}{(1)_{(a-2k-2)/2}}\right]\end{aligned}\tag{4.8}$$

if a is even. If we remember the assertion in Lemma 3 and the factors we have taken out of the determinant in (4.3) and (4.5), and if we then specialise $x=(a+b)/2$ and $y=(a+c-1)/2$, a tedious but straightforward calculation yields that in order to complete a proof of Lemma 3 it remains to show (4.7) and (4.8).

As described in the outline of the proof, we first prove that the claimed linear factors, which are

$$\left(\prod_{i=2}^{a-1}(1+b+c)_{i-1}\right)\left(\frac{b+1}{2}\right)^2_{(a-1)/2}\left(\frac{c+2}{2}\right)_{(a-1)/2}\left(\frac{1+b+c}{2}\right)_{(a-1)/2}2^{2a-2}\tag{4.9}$$

[2] In fact, only the linear factors in front of the sums in (4.7) and (4.8) can immediately be guessed, by computing $\det(D_a(b,c))$ for $a=2,3,\dots,8$. In order to work out an conjecture concerning the two sums themselves I first computed them for $a=2,3,\dots,23$. On the basis of these data, I worked out (guesses for) 'nice' evaluations of these polynomials at special values of b, for *all* values of a. (See the Outline of the proof of Lemma 3 and Step 3 of the proof of Lemma 3 for more details, and, in particular, for an explicit listing of these special values of b). Thereby I was helped by Krattenthaler's *Mathematica* 'guessing machine' RATE (which is available via Internet at `http://radon.mat.univie.ac.at/People/kratt/rate/rate.html`; see also [9, Appendix A]). Once I had got thus far, I computed the polynomials in (4.7) and (4.8) by Lagrange interpolation.

if a is odd, and, respectively,

$$\left(\prod_{i=2}^{a-1}(1+b+c)_{i-1}\right) b\left(\frac{b+2}{2}\right)^2_{(a-2)/2}\left(\frac{c+1}{2}\right)_{a/2}\left(\frac{1+b+c}{2}\right)_{a/2} 2^{2a-2} \tag{4.10}$$

if a is even, divide $\det(D_a(b,c))$ (see Step 2). In Step 3 we finally calculate the remaining irreducible polynomial in b and c (over $\mathbb{Z}$), which reads

$$\left(\prod_{i=2}^{a}(i-1)!\right)\left(\frac{1}{2}\right)^{a-1}\sum_{k=0}^{(a-1)/2}\left[\left(\frac{c+1}{2}\right)_k\left(\frac{1+b+c}{2}\right)_k\left(\frac{c+2k+2}{2}\right)_{(a-2k-1)/2}\right.$$
$$\left.\times\left(\frac{b+c+2k+3}{2}\right)_{(a-2k-1)/2}\frac{(\frac{1}{2})_{(a-2k-1)/2}}{(1)_{(a-2k-1)/2}}\right] \tag{4.11}$$

if a is odd, and respectively,

$$\left(\prod_{i=2}^{a}(i-1)!\right)\left(\frac{1}{2}\right)^{a}\sum_{k=0}^{(a-2)/2}\left[\left(\frac{c+2}{2}\right)_k\left(\frac{1+b+c}{2}\right)_k\left(\frac{c+2k+3}{2}\right)_{(a-2k-2)/2}\right.$$
$$\left.\times\left(\frac{b+c+2k+3}{2}\right)_{(a-2k-2)/2}\frac{(\frac{1}{2})_{(a-2k-2)/2}}{(1)_{(a-2k-2)/2}}\right] \tag{4.12}$$

if a is even.

Step 2: The linear factors in (4.9), respectively in (4.10), divide $\det(D_a(b,c))$ as a polynomial in b and c.

Essentially we have four different types of linear factors:

1. Factors of the form $(c+k)$: If a is odd, $k=2,4,\dots,a-1$. Otherwise $k=1,3,\dots,a-1$.
2. Factors of the form $(b+k)^2$: If a is odd, $k=1,3,\dots,a-2$. Otherwise $k=2,4,\dots,a-2$. The factor b occurs once if a even.
3. Factors of the form $(b+c+k)^{a-k}$ for $k=1,3,\dots,2\lceil(a-1)/2\rceil-1$.
4. Factors of the form $(b+c+k)^{a-k-1}$ for $k=2,4,\dots,2\lfloor(a-1)/2\rfloor$.

re 1. — The factors of the form $(c+k)$ *divide* $\det(D_a(b,c))$*:* In order to show that $(c+k)$ is a linear factor of $\det(D_a(b,c))$, where $1\le k\le a$ and $k\not\equiv a \pmod 2$, it suffices to show that $\det(D_a(b,-k))=0$ for those special k's. Therefore we search for linearly dependent rows or columns in the matrices $D_a(b,-k)$.

By computer experiments [3], I found the following non-trivial linear combination of rows:

$$\sum_{i=(a-k+3)/2}^{a-k+1}\frac{(-1)^i(b+i)_{a-k+1-i}\left(\frac{-a+k-1+2i}{2}\right)_{a-k+1-i}}{(1)_{a-k+1-i}\left(\frac{b-a+2i-2}{2}\right)_{a-k+1-i}}D_a(b,-k)_{(i,j)}=0, \tag{4.13}$$

for $j=2,3,\dots,a$.

In order to complete the proof that $(c+k)$ divides the determinant $\det(D_a(b,c))$, we just have to prove this identity. *Gosper's algorithm* [7] for hypergeometric sums (which are sums, where the quotient of two successive summands is a rational function in the summation index) helps us to recognize that this is actually a telescoping sum: First we examine the

[3]Again, I computed these linear combinations for small values of a, i.e., I solved the following system of linear equations in $c_i(a,b,k)$: $\sum_{i=2}^{a}c_i(a,b,k)D_a(b,-k)_{(i,j)}=0$, $j=2,3,\dots,a$, for $a=2,3,\dots,15$. I was led to an conjecture for the coefficient $c_i(a,b,k)$ for all values of a by using Krattenthaler's *Mathematica* 'guessing machine' RATE (which is available via Internet at `http://radon.mat.univie.ac.at/People/kratt/rate/rate.html`).

factor $((-a+k-1+2i)/2)_{a-k+1-i}$ in the linear combination. I claim that this factor is equal to zero if $i<(a-k+3)/2$ and because of that we can omit the lower bound on the summation index on the left-hand side of (4.13): In this case $(-a+k-1+2i)/2 \le 0$ and $(a-k-1)/2 \ge 0$. Because of that, and since $(-a+k-1+2i)/2$ is an integer, one of the factors of the Pochhammer symbol

$$\left(\frac{-a+k-1+2i}{2}\right)_{a-k+1-i} = \left(\frac{-a+k-1+2i}{2}\right)\left(\frac{-a+k-1+2i}{2}+1\right)\dots\left(\frac{a-k-1}{2}\right)$$

is equal to zero. If we then reverse the order of summation, we obtain that we must show the following identity:

$$\sum_{i=0}^{\infty} \frac{(-1)^j\left(\frac{1}{2}-\frac{a}{2}+\frac{k}{2}\right)_i(-a-b+k)_{i+j-2}}{(1)_{-a+i+j}\left(1-\frac{a}{2}-\frac{b}{2}+k\right)_i} H\left|_{x=(a+b)/2,y=(a+c-1)/2,c=-k,i\to a-k+1-i}\right. = 0. \tag{4.14}$$

Let $f(i)$ denote the summand of the sum in (4.14). The magnificent algorithm due to Gosper decides whether there exists a hypergeometric $g(i)$ with

$$g(i+1)-g(i)=f(i).$$

And in case of its existence the algorithm also computes $g(i)$. If such an $g(i)$ was found we would have

$$\sum_{i=0}^{\infty} f(i) = \sum_{i=0}^{\infty}(g(i+1)-g(i)) = \lim_{i\to\infty}(g(i)-g(0)).$$

($\lim_{i\to\infty} g(i)$ exists in our cases, since $f(i)=0$ for all but a finite number of i and therefore $g(i)$ is finally constant.)

A computer implementation of Gosper's algorithm [12] prints out

$$\frac{(-1)^j(-a-b+k)_{i+j-2}\left(\frac{1}{2}-\frac{a}{2}+\frac{k}{2}\right)_i(-1-b+k)(i-j+k)}{2(1)_{-1-a+i+j}\left(1-\frac{a}{2}-\frac{b}{2}+k\right)_{i-1}}$$

to be a suitable $g(i)$ for our $f(i)$. One may check the identity $g(i+1)-g(i)=f(i)$ by dividing the left-hand side by the right-hand side and simplifying the resulting rational function to 1.

This implies $\sum\limits_{i=0}^{\infty} f(i)=0$, i.e., the truth of (4.14), since $g(0)=0$, caused by the factor $1/(1)_{-1-a+j}$, and $g(i)=0$ for $i\ge(1+a-k)/2$, caused by $((1+a-k-2i)/2)_i$ (again we use the fact that $1\le k\le a$ and $k\not\equiv a \pmod 2$). Thus we have proved that $c+k$ is a factor of $\det(D_a(b,c))$.

re 2. — The factors of the form $(b+k)$ divide $\det(D_a(b,c))$*:* Next we have to prove that $(b+k)^2$ is a factor of $\det(D_a(b,c))$ if $0<k<a$ and $k\equiv a \pmod 2$, and, in addition, that b is a factor of $\det(D_a(b,c))$ if a is even.

In case that $k=a-2$, this is easy: If we examine the matrix $D_a(-a+2,c)$ carefully, we observe that the $(a-1)$th and ath row vanish. (For $j>2$ this is caused by $(b+i-j+2)_{j-2}$, and for $j=2$ by the nasty polynomial H.) Therefore $(b+a-2)$ is a factor of the $(a-1)$th row and a factor of the ath row of $D_a(b,c)$. Thus, we have proved that $(b+a-2)^2$ is a factor of the determinant $\det(D_a(b,c))$.

In case that $k<a-2$ we use the same method as for the factors of form $(c+k)$ above. But since we are now dealing with factors of higher multiplicity, it is not enough to find just one linear combination of rows or columns of $D_a(-k,c)$. In fact we have to find two linearly independent combinations if we want to prove that $(b+k)^2$ is a factor.

We claim that

$$\sum_{i=k+2}^{(a+k+2)/2} \frac{(-1)^{i-k}(c+a-i+2)_{i-k-2}\left(\frac{a+k-2i+4}{2}\right)_{i-k-2}}{(1)_{i-k-1}\left(\frac{c+a-2i+3}{2}\right)_{i-k-2}^2} \\ \times p_{i-k-1}(c+a-k-i+1)D_a(-k,c)_{(i,j)}=0 \quad (4.15)$$

if $0\le k<a-2$ and $k\equiv a\pmod 2$, and that

$$\sum_{i=k+3}^{(a+k+2)/2} \frac{(-1)^{i-k}(c+a-i+2)_{i-k-1}\left(\frac{a+k-2i+4}{2}\right)_{i-k-1}}{(1)_{i-k-1}\left(\frac{c+a-2i+3}{2}\right)_{i-k-2}^2} \\ \times p_{i-k-2}(c+a-k-i)D_a(-k,c)_{(i,j)}=-D_a(-k,c)_{(k+1,j)} \quad (4.16)$$

if $0<k<a-2$ and $k\equiv a\pmod 2$ are such linearly independent linear combinations of rows, $p_n(c)$ being the sequence of polynomials given by

$$p_n(c)=\sum_{h=0}^{n-1}\left(\frac{1+c-n}{2}\right)_{n-h-1}\left(\frac{1+c-2h+n}{2}\right)_h. \quad (4.17)$$

Notice that the exceptional factor b of $\det(D_a(b,c))$ for a is even, is included in the first linear combination (4.15).

In order to see that the two linear combinations in (4.15) and (4.16) are linearly independent we remark that the first linear combination (4.15) involves the $(k+1)$th row of $D_a(-k,c)$ whereas the second linear combination (4.16) does not (here we use $k<a-2$).

It remains to show the two hypergeometric identities (4.15) and (4.16). The situation is a bit more complicated this time compared to the hypergeometric identity (4.13) that proved that $(c+k)$ divides $\det(D_a(b,c))$: Actually these two identities are double sum identities, since they involve the polynomials $p_n(c)$. Luckily the proofs of the two identities are quite similar. We start with (4.15).

To begin with, we split the sum into four smaller sums — each one corresponding to one of the four summands of H (according to the representation of H in (4.4)):

$$\sum_{i=k+2}^{(a+k+2)/2} \frac{(-1)^{i-k-2}(c+a-i+2)_{i-k-2}\left(\frac{a+k-2i+4}{2}\right)_{i-k-2}}{(1)_{i-k-1}\left(\frac{c+a-2i+3}{2}\right)^2_{i-k-2}} \times p_{i-k-1}(c+a-k-i+1) D_a(-k,c)_{(i,j)} =$$

$$\sum_{i=k+2}^{(a+k+2)/2} \frac{(-1)^{i-k-2}(c+a-i+2)_{i-k-2}\left(\frac{a+k-2i+4}{2}\right)_{i-k-2}}{(1)_{i-k-1}\left(\frac{c+a-2i+3}{2}\right)^2_{i-k-2}} p_{i-k-1}(c+a-k-i+1)$$
$$\times (-k+i-j+1)_{j-1}(c-i+j+1)_{a-j+1}((-1-a+c+2j)/2)((-2-a+2i-k)/2)$$
$$+ \sum_{i=k+2}^{(a+k+2)/2} \frac{(-1)^{i-k-2}(c+a-i+2)_{i-k-2}\left(\frac{a+k-2i+4}{2}\right)_{i-k-2}}{(1)_{i-k-1}\left(\frac{c+a-2i+3}{2}\right)^2_{i-k-2}} p_{i-k-1}(c+a-k-i+1)$$
$$\times (-k+i-j+2)_{j-2}(c-i+j)_{a-j+2}((-2-a+2i-k)/2)((2+a-2j-k)/2)$$
$$+ \sum_{i=k+2}^{(a+k+2)/2} \frac{(-1)^{i-k-2}(c+a-i+2)_{i-k-2}\left(\frac{a+k-2i+4}{2}\right)_{i-k-2}}{(1)_{i-k-1}\left(\frac{c+a-2i+3}{2}\right)^2_{i-k-2}} p_{i-k-1}(c+a-k-i+1)$$
$$\times (-k+i-j)_{j}(c-i+j+2)_{a-j}((3+a+c-2i)/2)((-1-a+c+2j)/2)$$
$$+ \sum_{i=k+2}^{(a+k+2)/2} \frac{(-1)^{i-k-2}(c+a-i+2)_{i-k-2}\left(\frac{a+k-2i+4}{2}\right)_{i-k-2}}{(1)_{i-k-1}\left(\frac{c+a-2i+3}{2}\right)^2_{i-k-2}} p_{i-k-1}(c+a-k-i+1)$$
$$\times (-k+i-j+1)_{j-1}(c-i+j+1)_{a-j+1}((3+a+c-2i)/2)((2+a-2j-k)/2). \quad (4.18)$$

Next we examine the sum of the first and the third sum of the right-hand side in (4.18). After some simplifications and interchange of summations one gets

$$\sum_{h=0}^{(a-k-2)/2} \left[\sum_{i=h+k+2}^{(a+k+2)/2} (-1)^{a-j+1} \frac{\left(-\frac{1}{2}-\frac{a}{2}+\frac{c}{2}+j\right)\left(\frac{1}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_h}{\left(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_h} \times \frac{\left(1-\frac{a}{2}+\frac{k}{2}\right)_{-1+i-k}(1-a-c+k)_{-1+a+i-j-k}}{(1)_{i-j-k}\left(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_{-2+i-k}} \right]$$
$$- \sum_{h=0}^{(a-k-2)/2} \left[\sum_{i=h+k+2}^{(a+k+2)/2} (-1)^{a-j+1} \frac{\left(-\frac{1}{2}-\frac{a}{2}+\frac{c}{2}+j\right)\left(\frac{1}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_h}{\left(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_h} \times \frac{\left(1-\frac{a}{2}+\frac{k}{2}\right)_{-2+i-k}(1-a-c+k)_{-2+a+i-j-k}}{(1)_{-1+i-j-k}\left(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_{-3+i-k}} \right].$$

Notice that these two double sums are quite similar. If we combine the exterior sums, take some common factors out of the two inner sum and shift the index of the first inner sum by one, we obtain

$$\sum_{h=0}^{(a-k-2)/2} (-1)^{a-j+1} \frac{\left(-\frac{1}{2}-\frac{a}{2}+\frac{c}{2}+j\right)\left(\frac{1}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_h}{\left(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_h}$$
$$\left[\sum_{i=h+k+3}^{(a+k+4)/2} \frac{\left(1-\frac{a}{2}+\frac{k}{2}\right)_{-2+i-k}(1-a-c+k)_{-2+a+i-j-k}}{(1)_{-1+i-j-k}\left(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_{-3+i-k}}\right.$$
$$\left.-\sum_{i=h+k+2}^{(a+k+2)/2} \frac{\left(1-\frac{a}{2}+\frac{k}{2}\right)_{-2+i-k}(1-a-c+k)_{-2+a+i-j-k}}{(1)_{-1+i-j-k}\left(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_{-3+i-k}}\right].$$

This makes us discover, that the two inner sums nearly cancel out each other, because the two summands are exactly the same. Therefore the latter expression simplifies further to

$$\sum_{h=0}^{(a-k-2)/2} (-1)^{a-j+1} \frac{\left(-\frac{1}{2}-\frac{a}{2}+\frac{c}{2}+j\right)\left(\frac{1}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_h}{\left(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_h}$$
$$\times \frac{\left(1-\frac{a}{2}+\frac{k}{2}\right)_h (1-a-c+k)_{a+h-j}}{(1)_{1+h-j}\left(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_{-1+h}}.$$

Thus we have already reduced the sum of the two double sums to the following single sum

$$\sum_{h=0}^{(a-k-2)/2} (-1)^{a-j} \frac{\left(-\frac{1}{2}-\frac{a}{2}+\frac{c}{2}+j\right)\left(\frac{1}{2}-\frac{a}{2}-\frac{c}{2}+k\right)\left(1-\frac{a}{2}+\frac{k}{2}\right)_h (1-a-c+k)_{a+h-j}}{\left(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_h (1)_{1+h-j}}. \tag{4.19}$$

Analogously one can show that the sum of the second und the fourth sum in (4.18), which reads

$$\sum_{h=0}^{(a-k-2)/2} \sum_{i=2+h+k}^{(a+k+2)/2} (-1)^{-1+a-j} \frac{\left(1+\frac{a}{2}-j-\frac{k}{2}\right)\left(\frac{1}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_h}{\left(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_h}$$
$$\times \frac{\left(1-\frac{a}{2}+\frac{k}{2}\right)_{-1+i-k}(1-a-c+k)_{a+i-j-k}}{(1)_{1+i-j-k}\left(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_{-2+i-k}}$$
$$-\sum_{h=0}^{(a-k-2)/2} \sum_{i=2+h+k}^{(a+k+2)/2} (-1)^{-1+a-j} \frac{\left(1+\frac{a}{2}-j-\frac{k}{2}\right)\left(\frac{1}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_h}{\left(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_h}$$
$$\times \frac{\left(1-\frac{a}{2}+\frac{k}{2}\right)_{-2+i-k}(1-a-c+k)_{-1+a+i-j-k}}{(1)_{i-j-k}\left(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_{-3+i-k}},$$

after interchange of summation and some cancellations, simplifies to

$$\sum_{h=0}^{(a-k-2)/2} (-1)^{a-j} \frac{\left(1+\frac{a}{2}-j-\frac{k}{2}\right)\left(\frac{1}{2}-\frac{a}{2}-\frac{c}{2}+k\right)\left(1-\frac{a}{2}+\frac{k}{2}\right)_h (1-a-c+k)_{1+a+h-j}}{(1)_{2+h-j}\left(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_h}. \tag{4.20}$$

Now it remains to show that the two (single) sums (4.19) and (4.20) sum up to zero. Because of the factor $1/(1)_{1+h-j}$, we are able to change the lower bound on the summation index of the sum in (4.19) to $j-1$, since $j \geq 2$ and $1/(1)_n = 0$ if n is a negative integer.

Analogously we change the lower bound of the summation index of the sum in (4.20) to $j-2$ according to the factor $1/(1)_{2+h-j}$. Therefore the sum of (4.19) and (4.20) is

$$\left(\frac{1}{2}-\frac{a}{2}-\frac{c}{2}+k\right) \times \left[\sum_{h=j-1}^{(a-k-2)/2} (-1)^{a-j} \frac{\left(-\frac{1}{2}-\frac{a}{2}+\frac{c}{2}+j\right)\left(1-\frac{a}{2}+\frac{k}{2}\right)_h (1-a-c+k)_{a+h-j}}{\left(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_h (1)_{1+h-j}} + \sum_{h=j-2}^{(a-k-2)/2} (-1)^{a-j} \frac{\left(1+\frac{a}{2}-j-\frac{k}{2}\right)\left(1-\frac{a}{2}+\frac{k}{2}\right)_h (1-a-c+k)_{1+a+h-j}}{(1)_{2+h-j}\left(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_h}\right].$$

Using the standard hypergeometric notation

$${}_rF_s\left[\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};z\right]=\sum_{k=0}^{\infty}\frac{(a_1)_k\dots(a_r)_k}{k!(b_1)_k\dots(b_s)_k}z^k,$$

and, after canceling the factor $\left(\frac{1}{2}-\frac{a}{2}-\frac{c}{2}+k\right)$, it remains to show that

$$\frac{\left(1-\frac{a}{2}+\frac{k}{2}\right)_{-1+j}(1-a-c+k)_{-1+a}}{\left(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+k\right)_{-1+j}} \times \left({}_2F_1\left[\begin{matrix}-\frac{a}{2}+j+\frac{k}{2},-c+k\\ \frac{1}{2}-\frac{a}{2}-\frac{c}{2}+j+k\end{matrix};1\right]\left(-\frac{1}{2}-\frac{a}{2}+\frac{c}{2}+j\right) - {}_2F_1\left[\begin{matrix}-1-\frac{a}{2}+j+\frac{k}{2},-c+k\\ -\frac{1}{2}-\frac{a}{2}-\frac{c}{2}+j+k\end{matrix};1\right]\left(-\frac{1}{2}-\frac{a}{2}-\frac{c}{2}+j+k\right)\right)=0. \quad (4.21)$$

The factor $((2-a+k)/2)_{-1+j}$ tells us that this is true at least for $j\geq(-2+a-k)/2$. Otherwise, we use Vandermonde's summation formula (see [13, (1.7.7), Appendix (III.4)])

$${}_2F_1\left[\begin{matrix}a,-n\\ c\end{matrix};1\right]=\frac{(c-a)_n}{(c)_n}, \quad (4.22)$$

which is valid if n is a nonnegative integer. Namely, if we apply this formula to the left-hand side of (4.21) with $n=(a-2j-k)/2$, respectively $n=(2+a-2j-k)/2$ (here we use $j<(-2+a-k)/2$), we see that the two ${}_2F_1$–series in (4.21) sum up to zero. Thus, we have proved the first linear combination, (4.15), for the factors of form $(b+k)$.

The proof of the second linear combination (4.16) is quite similar: Again we split the double sum into four smaller sums according to the summands of the polynomial H. After some simplifications and interchange of summation we have reduced our problem to the following hypergeometric identity:

$$\sum_{h=0}^{(-4+a-k)/2}\sum_{i=3+h+k}^{(2+a+k)/2}\frac{(-1)^{1+a+h-j}\,(-\frac{1}{2}-\frac{a}{2}+\frac{c}{2}+j)_1\,(\frac{-a}{2}+\frac{k}{2})_{i-k}\,(-a-c+k)_{a+i-j-k}}{(1)_{i-j-k}\,(-\frac{3}{2}+\frac{a}{2}+\frac{c}{2}-h-k)_{1+h}\,(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+h+k)_{-2-h+i-k}}$$
$$+\sum_{h=0}^{(-4+a-k)/2}\sum_{i=3+h+k}^{(2+a+k)/2}\frac{(-1)^{1+a+h-j}\,(1+\frac{a}{2}-j-\frac{k}{2})_1\,(\frac{-a}{2}+\frac{k}{2})_{i-k}\,(-a-c+k)_{1+a+i-j-k}}{(1)_{1+i-j-k}\,(-\frac{3}{2}+\frac{a}{2}+\frac{c}{2}-h-k)_{1+h}\,(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+h+k)_{-2-h+i-k}}$$
$$+\sum_{h=0}^{(-4+a-k)/2}\sum_{i=3+h+k}^{(2+a+k)/2}\frac{(-1)^{a+h-j}\,(-\frac{1}{2}-\frac{a}{2}+\frac{c}{2}+j)_1\,(\frac{-a}{2}+\frac{k}{2})_{-1+i-k}\,(-a-c+k)_{-1+a+i-j-k}}{(1)_{-1+i-j-k}\,(-\frac{3}{2}+\frac{a}{2}+\frac{c}{2}-h-k)_{1+h}\,(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+h+k)_{-3-h+i-k}}$$
$$+\sum_{h=0}^{(-4+a-k)/2}\sum_{i=3+h+k}^{(2+a+k)/2}\frac{(-1)^{a+h-j}\,(1+\frac{a}{2}-j-\frac{k}{2})_1\,(\frac{-a}{2}+\frac{k}{2})_{-1+i-k}\,(-a-c+k)_{a+i-j-k}}{(1)_{i-j-k}\,(-\frac{3}{2}+\frac{a}{2}+\frac{c}{2}-h-k)_{1+h}\,(\frac{3}{2}-\frac{a}{2}-\frac{c}{2}+h+k)_{-3-h+i-k}}$$
$$=-D_a(-k,c)_{(k+1,j)}. \quad (4.23)$$

As before, the sum of the first and the third summand on the left-hand side of (4.23) simplifies to a single sum:

$$\sum_{h=0}^{(-4+a-k)/2}(-1)^{a+h-j}\frac{\left(-\frac{1}{2}-\frac{a}{2}+\frac{c}{2}+j\right)_1\left(-\frac{a}{2}+\frac{k}{2}\right)_{2+h}(-a-c+k)_{2+a+h-j}}{\left(-\frac{3}{2}+\frac{a}{2}+\frac{c}{2}-h-k\right)_{1+h}(1)_{2+h-j}}. \quad (4.24)$$

Analogously, the second and fourth summand of (4.23) give

$$\sum_{h=0}^{(-4+a-k)/2}(-1)^{a+h-j}\frac{\left(1+\frac{a}{2}-j-\frac{k}{2}\right)_1\left(-\frac{a}{2}+\frac{k}{2}\right)_{2+h}(-a-c+k)_{3+a+h-j}}{\left(-\frac{3}{2}+\frac{a}{2}+\frac{c}{2}-h-k\right)_{1+h}(1)_{3+h-j}}. \quad (4.25)$$

Again we want to change the lower bound of the remaining single sums in (4.24) and (4.25) to $j-2$, respectively to $j-3$, according to the factor $1/(1)_{2+h-j}$, respectively $1/(1)_{3+h-j}$. This is because then the two sums are ${}_2F_1$–series and this makes it possible to apply Vandermonde's summation formula (4.22) again. In case of (4.25) and $j=2$ this change is problematic since $j-3<0$ for $j=2$. But the right-hand side of (4.16) compensates this missing term.

In terms of hypergeometric notation, it remains to show that

$$\frac{(-1)^a\,(\frac{-a}{2}+\frac{k}{2})_j\,(-a-c+k)_a}{(\frac{1}{2}+\frac{a}{2}+\frac{c}{2}-j-k)\,(\frac{3}{2}+\frac{a}{2}+\frac{c}{2}-j-k)_{-2+j}}$$
$$\times\left[{}_2F_1\left[\begin{matrix}-\frac{a}{2}+j+\frac{k}{2},-c+k\\ \frac{1}{2}-\frac{a}{2}-\frac{c}{2}+j+k\end{matrix};1\right]\left(-\frac{1}{2}-\frac{a}{2}+\frac{c}{2}+j\right)\right.$$
$$\left.+\,{}_2F_1\left[\begin{matrix}-1-\frac{a}{2}+j+\frac{k}{2},-c+k\\ -\frac{1}{2}-\frac{a}{2}-\frac{c}{2}+j+k\end{matrix};1\right]\left(\frac{1}{2}+\frac{a}{2}+\frac{c}{2}-j-k\right)\right]=0.$$

For $j\ge(a-k+1)/2$ this is obvious because of the factor $((-a+k)/2)_j$. Otherwise, we use again Vandermonde's summation formula (4.22) with $n=(a-2j-k)/2$, respectively $n=(2+a-2j-k)/2$.

Finally we have completely proved that $(b+k)^2$ is a factor of $\det(D_a(b,c))$.

re 3., 4. — $\prod_{i=2}^{a}(b+c+1)_{i-1}/((b+c+2)/2)_{\lceil(a-2)/2\rceil}$ is a factor of $\det(D_a(b,c))$*:* In order to prove that the third and the fourth type of factors (see the beginning of Step 2) divide $\det(D_a(b,c))$, we could use the same procedure as for the factors of form $(c+k)$ and the factors of form $(b+k)^2$. But, fortunately, we will see that the triple sum that is the subject of

Section 3 provides an easier proof. 'Fortunately,' since in this case the procedure leads to quite complicated hypergeometric identities. Furthermore, the underlying linear combinations are more difficult to guess because an additional parameter originating in the higher multiplicity of the factors is involved.

If we look at (4.3) and (4.5), we see that (4.6) is equal to (3.1) evaluated at $x = (a+b)/2$ and $y = (a+c-1)/2$ modulo some factors. Therefore a combination of (4.3), (4.5) and Lemma 2 gives

$$\det(D_a(b,c)) = \left(\prod_{i=2}^{a} (b+c+1)_{i-1}(i-1)!\right) \left(\frac{c-a+3}{2}\right)_{a-1} \left(\frac{b-a+2}{2}\right)_{a-1}$$
$$\times \sum_{n=1}^{a} \sum_{m=1}^{a} \sum_{s=1}^{m} \left[(-1)^{n+s} \frac{\left(\frac{1-a+c+2n}{2}\right)_{(a+b-2)/2}}{\left(\frac{3-a+c}{2}\right)_{(a+b-2)/2}} \frac{\left(\frac{-a+b+2s}{2}\right)_{(a+c-1)/2}}{\left(\frac{2-a+b}{2}\right)_{(a+c-1)/2}} \binom{m-1}{s-1} \right.$$
$$\left. \frac{(b+1)_{s-1}}{(b+c+1)_{s-1}} \frac{(c+1)_{n-1}}{(n-1)!} \frac{(b+c+n)_{m-n}}{(m-n)!} \right]. \tag{4.26}$$

We are aiming to show that $\prod_{i=2}^{a}(b+c+1)_{i-1}/((b+c+2)/2)_{\lceil (a-2)/2 \rceil}$ is a factor of (4.6). Using (4.26), it remains to show that the following sum is equal to a polynomial in b and c for all integers b and c if we fix a:

$$\left(\prod_{i=2}^{a} (i-1)!\right) \left(\frac{b+c+2}{2}\right)_{\lceil (a-2)/2 \rceil} \left(\frac{c-a+3}{2}\right)_{a-1} \left(\frac{b-a+2}{2}\right)_{a-1}$$
$$\times \sum_{n=1}^{a} \sum_{m=1}^{a} \sum_{s=1}^{m} \left[(-1)^{n+s} \frac{\left(\frac{1-a+c+2n}{2}\right)_{(a+b-2)/2}}{\left(\frac{3-a+c}{2}\right)_{(a+b-2)/2}} \frac{\left(\frac{-a+b+2s}{2}\right)_{(a+c-1)/2}}{\left(\frac{2-a+b}{2}\right)_{(a+c-1)/2}} \binom{m-1}{s-1} \right.$$
$$\left. \frac{(b+1)_{s-1}}{(b+c+1)_{s-1}} \frac{(c+1)_{n-1}}{(n-1)!} \frac{(b+c+n)_{m-n}}{(m-n)!} \right]. \tag{4.27}$$

As it stands, this does not appear as a polynomial in b and c, since b and c occur in the second arguments of some Pochhammer symbols. But by using

$$\frac{(r)_n}{(s)_n} = \frac{(s+n)_{r-s}}{(s)_{r-s}}, \tag{4.28}$$

which is valid for all integers n, we are able to ban b and c from the second arguments of the Pochhammer symbols in (4.27). Then, after some cancellations, we obtain

$$\prod_{i=2}^{a} (i-1)! \sum_{n=1}^{a} \sum_{m=1}^{a} \sum_{s=1}^{m} (-1)^{n+s} \left(\frac{1-a+c+2n}{2}\right)_{a-n} \left(\frac{b+c+1}{2}\right)_{n-1} \left(\frac{-a+b+2s}{2}\right)_{a-s}$$
$$\times \binom{m-1}{s-1} \frac{\left(\frac{b+c+2}{2}\right)_{\lceil (a-2)/2 \rceil} \left(\frac{b+c+1}{2}\right)_{s-1}}{(b+c+1)_{s-1}} \frac{(b+1)_{s-1}(c+1)_{n-1}(b+c+n)_{m-n}}{(n-1)!(m-n)!}$$

for (4.27), and this is manifestly a polynomial in b and c. Indeed, because of $s \le m \le a$, thanks to the binomial coefficient, the term $(b+c+1)_{s-1}$ cancels with the two terms on top of the fraction.

Therefore we have finally proved that the linear factors in (4.9), respectively in (4.10), divide the determinant $\det(D_a(b,c))$ as a polynomial in b and c, and we may now turn to the computation of the remaining irreducible polynomial.

Step 3: Computation of the irreducible polynomial. We emphasize once more that Step 1 and Step 2 show that

$$det\,(D_a(b,c)) = (\text{Linear factors in (4.9), respectively (4.10)}) \times P_a(b,c), \tag{4.29}$$

where $P_a(b,c)$ is a certain polynomial in b and c if we fix a. In this final step we prove that $P_a(b,c)$ equals (4.11), respectively (4.12).

The computation of the irreducible polynomial, which will be denoted by $P_a(b,c)$, is done in the following way: We consider the polynomial $P_a(b,c)$ as a polynomial in b over $\mathbb{Z}[c]$ and find that it has 'nice' evaluations at $b=-c-k$, k odd and $1 \le k \le a$, see (4.30) and (4.31). We are able to prove these 'nice' evaluations with the help of Lemma 2. Furthermore, we will see that $P_a(b,c)$ is a polynomial in b with a degree smaller or equal to $\lfloor (a-1)/2 \rfloor$. Therefore these 'nice' evaluations are just enough so that we can compute $P_a(b,c)$ by using Lagrange's interpolation formula, see (4.32), respectively (4.33).

First we will convince ourselves that the assertion about the degree of $P_a(b,c)$ in b is true: The entry in the ith row and the jth column of the matrix $D_a(b,c)$, see (4.6), is a polynomial in b of degree $j+1$, since H is a polynomial in b of degree 2. Hence, in the defining expansion of the determinant, each summand has degree $2+3+\cdots+a = (a+2)(a-1)/2$ in b and therefore the degree of the determinant itself is at most $(a+2)(a-1)/2$. One easily checks that the product of the linear factors is a polynomial in b with degree $a^2/2$ in case of a is even and $(a+1)(a-1)/2$ otherwise. Because of that $(a+2)(a-1)/2 - a^2/2 = \lfloor (a-1)/2 \rfloor$ respectively $(a+2)(a-1)/2 - (a+1)(a-1)/2 = \lfloor (a-1)/2 \rfloor$ is an upper estimation for the degree of $P_a(b,c)$ in b. Later we will see that this is in fact the exact degree.

We claim that

$$\begin{aligned} P_a(-c-k,c) = \left(\frac{1}{2}\right)^{a-1} \left(\prod_{i=2}^{a} (i-1)!\right) \left(\frac{1}{2}\right)_{(a-1)/2} \frac{\left(\frac{k-1}{2}\right)!(-1)^{(k-1)/2}}{\left(\frac{a-k+1}{2}\right)_{(k-1)/2}} \\ \times \left(\frac{c+k+1}{2}\right)_{(a-k)/2} \left(\frac{c+1}{2}\right)_{(k-1)/2} \end{aligned} \tag{4.30}$$

if a is odd and $k=1,3,\dots,a$, and

$$\begin{aligned} P_a(-c-k,c) = \left(\frac{1}{2}\right)^{a} \left(\prod_{i=2}^{a} (i-1)!\right) \left(\frac{1}{2}\right)_{(a-2)/2} \frac{\left(\frac{k-1}{2}\right)!(-1)^{(k-1)/2}}{\left(\frac{a-k}{2}\right)_{(k-1)/2}} \\ \times \left(\frac{c+k+2}{2}\right)_{(a-k-1)/2} \left(\frac{c+2}{2}\right)_{(k-1)/2} \end{aligned} \tag{4.31}$$

if a is even and $k=1,3,\dots,a-1$.

Assuming the truth of the claim, we would have

$$\begin{aligned} P_a(b,c) = \left(\frac{1}{2}\right)^{a-1} \prod_{i=2}^{a} (i-1)! \sum_{k=0}^{(a-1)/2} \Bigg[\left(\frac{c+1}{2}\right)_k \left(\frac{1+b+c}{2}\right)_k \left(\frac{c+2k+2}{2}\right)_{(a-2k-1)/2} \\ \times \left(\frac{b+c+2k+3}{2}\right)_{(a-2k-1)/2} \frac{\left(\frac{1}{2}\right)_{(a-2k-1)/2}}{(1)_{(a-2k-1)/2}} \Bigg] \end{aligned} \tag{4.32}$$

if a is odd, and, respectively,

$$P_a(b,c) = \left(\frac{1}{2}\right)^a \prod_{i=2}^{a} (i-1)! \sum_{k=0}^{(a-2)/2} \left[\left(\frac{c+2}{2}\right)_k \left(\frac{1+b+c}{2}\right)_k \left(\frac{c+2k+3}{2}\right)_{(a-2k-2)/2} \right.$$
$$\left. \times \left(\frac{b+c+2k+3}{2}\right)_{(a-2k-2)/2} \frac{(\frac{1}{2})_{(a-2k-2)/2}}{(1)_{(a-2k-2)/2}} \right] \quad (4.33)$$

if a is even, by Lagrange interpolation.

Thus, it remains to show (4.30) and (4.31). By definition, $P_a(b,c)$ is the quotient of $\det(D_a(b,c))$ and the linear factors in (4.9), respectively in (4.10) (see (4.29)). This, together with a tedious but straightforward calculation shows that the claims in (4.30) and (4.31) conflate to the following

$$\left. \frac{\det(D_a(b,c)) \binom{\frac{b+c-1}{2}}{\frac{a+b-2}{2}} \binom{\frac{b+c-1}{2}}{\frac{a+c-1}{2}}}{\left(\prod_{i=2}^{a} (1+b+c)_{i-1} (i-1)!\right) \left(\frac{c-a+3}{2}\right)_{a-1} \left(\frac{b-a+2}{2}\right)_{a-1}} \right|_{b=-c-k} = (-1)^a \frac{(c+1)_{k-1}}{(k-1)!}. \quad (4.34)$$

The reader should notice that we are not able to directly set $b=-c-k$ on the left-hand side of (4.34), because the denominator vanishes for $b=-c-k$. As mentioned before we want to replace the determinant on the left-hand sides of (4.34) by the triple sum in Lemma 2. Still even after cancellations we are not able to directly set $b=-c-k$ in this triple sum, since then the denominator of the summand becomes zero if $k>n$.

In order to avoid an indefinite expression we act as follows: If we reexamine the proof of Lemma 2, we obtain the following generalization of Lemma 2:

Let a,b,c be positive integers and b',x,x be integers. Then the determinant

$$- \det_{1 \le i,j \le a+1} \begin{array}{c} \begin{array}{cc} 1 \le j \le a & j = a+1 \end{array} \\ \left(\begin{array}{c|c} \binom{b+c}{c-i+j} & \binom{b'-x+y}{y-i+1} \\ \hline \binom{c+x-y-1}{x-j} & 0 \end{array} \right) \end{array} \begin{array}{l} 1 \le i \le a \\ i = a+1 \end{array} \quad . \quad (4.35)$$

equals

$$\left(\prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}\right) \frac{(1)_c}{(b+1)_c}$$
$$\times \sum_{n=1}^{a} \sum_{m=1}^{a} \sum_{s=1}^{m} \left[(-1)^{n+s} \binom{c+x-y+n-2}{x-1} \binom{b'-x+y+s-1}{b'-x+s-1} \frac{(b+1)_{s-1}}{(b+c+1)_{s-1}} \right.$$
$$\left. \binom{m-1}{s-1} \frac{(c+1)_{n-1}}{(n-1)!} \frac{(b+c+n)_{m-n}}{(m-n)!} \right]. \quad (4.36)$$

We set $b=-c-k+\varepsilon$ and $b'=-c-k$ in (4.36). In view of the fact that (4.35) is continuous in b, and if we compare the determinant in (4.6) with the determinant in (2.6) evaluated at

$x=(a+b)/2$ and $y=(a+c-1)/2$, we obtain that the left-hand side of (4.34) is equal to

$$\lim_{\varepsilon\to 0}\sum_{n=1}^{a}\sum_{m=1}^{a}\sum_{s=1}^{m}\Bigg[(-1)^{n+s}\binom{\frac{-k+2n-3}{2}}{\frac{a-c-k-2}{2}}\binom{\frac{-k+2s-3}{2}}{\frac{a+c-1}{2}}\binom{m-1}{s-1}\\ \times\frac{(-c-k+\varepsilon+1)_{s-1}}{(-k+\varepsilon+1)_{s-1}}\frac{(c+1)_{n-1}}{(n-1)!}\frac{(-k+\varepsilon+n)_{m-n}}{(m-n)!}\Bigg]. \tag{4.37}$$

Now it remains to show that the right-hand side of (4.34) equals (4.37) for $k=1,3,\dots,2\lfloor a-1/2\rfloor$.

Therefore let us consider the triple sum (4.37). First of all, we are allowed to extend the sum over s to all positive integers, since $\binom{m-1}{s-1}=0$ if $s>m$. Using hypergeometric notation for this innermost sum, we get

$$\lim_{\varepsilon\to 0}\sum_{n=1}^{a}\sum_{m=1}^{a}\Bigg[(-1)^{n+1}\,{}_3F_2\left[\begin{matrix}1-m,1-c-k+\varepsilon,\frac{1}{2}-\frac{k}{2}\\ 1-k+\varepsilon,1-\frac{a}{2}-\frac{c}{2}-\frac{k}{2}\end{matrix};1\right]\\ \frac{(1+c)_{-1+n}\,(1-\frac{a}{2}-\frac{c}{2}-\frac{k}{2})_{-\frac{1}{2}+\frac{a}{2}+\frac{c}{2}}\,(\frac{1}{2}-\frac{a}{2}+\frac{c}{2}+n)_{-1+\frac{a}{2}-\frac{c}{2}-\frac{k}{2}}\,(-k+\varepsilon+n)_{m-n}}{(1)_{-\frac{1}{2}+\frac{a}{2}+\frac{c}{2}}\,(1)_{-1+\frac{a}{2}-\frac{c}{2}-\frac{k}{2}}\,(1)_{m-n}\,(1)_{-1+n}}\Bigg].$$

We apply the following transformation formula due to Thomae [5, (3.1.1)] to this hypergeometric sum,

$${}_3F_2\left[\begin{matrix}a,b,-n\\ d,e\end{matrix};1\right]=\frac{(-b+e)_n}{(e)_n}\,{}_3F_2\left[\begin{matrix}-n,b,-a+d\\ d,1+b-e-n\end{matrix};1\right] \tag{4.38}$$

with $a=1-m$, $b=1-c-k+\varepsilon$, $n=(k-1)/2$, $d=1-k+\varepsilon$ and $e=1-a/2-c/2-k/2$.

Writing the resulting ${}_3F_2$–series as a sum over s, after some simplifications and cancellations this gives

$$\lim_{\varepsilon\to 0}\sum_{n=1}^{a}\sum_{m=1}^{a}\sum_{s=0}^{\infty}\Bigg[(-1)^{\frac{1}{2}-\frac{3a}{2}+\frac{c}{2}+k+n+s}\left(-\frac{a}{2}+\frac{c}{2}+\frac{k}{2}\right)\\ \times\frac{(1+c)_{-1+n}\,(1-c-k+\varepsilon)_s\,(\frac{1}{2}-\frac{a}{2}+\frac{c}{2}+n)_{-1+\frac{a}{2}-\frac{c}{2}-\frac{k}{2}}\,(-k+\varepsilon+n)_{m-n+s}}{(1)_{m-n}\,(1)_{-1+n}\,(1)_{-\frac{1}{2}+\frac{k}{2}-s}\,(1)_s\,(1)_{\frac{1}{2}+\frac{a}{2}-\frac{c}{2}-k+s}\,(1-k+\varepsilon)_s}\Bigg].$$

Next we interchange the two inner sums and reverse the order of summation in the innermost sum. We obtain

$$\lim_{\varepsilon\to 0}\sum_{n=1}^{a}\sum_{s=0}^{\infty}\sum_{m=0}^{-1+a}\Bigg[(-1)^{\frac{1}{2}-\frac{3a}{2}+\frac{c}{2}+k+n+s}\left(-\frac{a}{2}+\frac{c}{2}+\frac{k}{2}\right)\\ \times\frac{(1+c)_{-1+n}\,(1-c-k+\varepsilon)_s\,(\frac{1}{2}-\frac{a}{2}+\frac{c}{2}+n)_{-1+\frac{a}{2}-\frac{c}{2}-\frac{k}{2}}\,(-k+\varepsilon+n)_{a-m-n+s}}{(1)_{a-m-n}\,(1)_{-1+n}\,(1)_{-\frac{1}{2}+\frac{k}{2}-s}\,(1)_s\,(1)_{\frac{1}{2}+\frac{a}{2}-\frac{c}{2}-k+s}\,(1-k+\varepsilon)_s}\Bigg].$$

The reason why reversing was so helpful, is that we now can extend the summation with respect to m to all nonnegative integers (since $1/(1)_{a-m-n}=0$ if $m>a-n$). If we use hypergeometric notation this yields

$$\lim_{\varepsilon\to 0}\sum_{n=1}^{a}\sum_{s=0}^{\infty}\Bigg[(-1)^{\frac{1}{2}-\frac{3a}{2}+\frac{c}{2}+k+n+s}\,{}_2F_1\left[{1,-a+n \atop 1-a+k+\varepsilon-s};1\right]$$
$$\times\frac{(1+c)_{-1+n}\,(1-c-k+\varepsilon)_s\,(\frac{-a}{2}+\frac{c}{2}+\frac{k}{2})\,(\frac{1}{2}-\frac{a}{2}+\frac{c}{2}+n)_{-1+\frac{a}{2}-\frac{c}{2}-\frac{k}{2}}\,(-k+\varepsilon+n)_{a-n+s}}{(1)_{a-n}\,(1)_{-1+n}\,(1)_{-\frac{1}{2}+\frac{k}{2}-s}\,(1)_s\,(1)_{\frac{1}{2}+\frac{a}{2}-\frac{c}{2}-k+s}\,(1-k+\varepsilon)_s}\Bigg].$$

We apply Vandermonde's summation formula (4.22) to this hypergeometric sum. After some manipulations this gives

$$\lim_{\varepsilon\to 0}\sum_{n=1}^{a}\sum_{s=0}^{\infty}\Bigg[(-1)^{-\frac{1}{2}+\frac{k}{2}}\left(-\frac{a}{2}+\frac{c}{2}+\frac{k}{2}\right)\frac{(-a+k-\varepsilon-s)_{a-n}\,(1+k-\varepsilon-n-s)_s}{(1)_{\frac{1}{2}+\frac{k}{2}-n}(1)_{-\frac{1}{2}+\frac{k}{2}-s}}$$
$$\times\frac{(1+c)_{-1+n}\,(1-c-k+\varepsilon)_s\,(\frac{3}{2}+\frac{a}{2}-\frac{c}{2}-k+s)_{-1+k-n-s}}{(1)_{a-n}\,(1)_{-1+n}\,(1)_s\,(1-k+\varepsilon)_s}\Bigg].$$

Now we are able to perform the limit $\varepsilon\to 0$, since, because of the factor $1/(1)_{-1/2+k/2-s}$, we have $k>s$. 'Performing the limit' means that we simply set $\varepsilon=0$.

With pleasure we notice that our triple sum has simplified to a double sum. Big surprise arises when I finally claim that the summand of the sum is only different from zero if and only if $n=k/2+1/2$ and $s=k/2-1/2$, and we therefore get rid of all sums: The factor $1/(1)_{1/2+k/2-n}$ implies $n\le 1/2+k/2$, and the factor $1/(1)_{-1/2+k/2-s}$ implies $s\le -1/2+k/2$. Next we notice that

$$(-a+k-s)_{a-n}(1+k-n-s)_s=\frac{(-a+k-s)_{1+a-n-s}}{(k-n-s)}.$$

The numerator on the right-hand side is zero since $-a+k-s\le -a+k\le 0$ and $k-n\ge 0$. Therefore the summand is only different from zero if $k-n-s=0$. This together with our first observation gives my assertion.

It remains to compute the summand in the last double sum for $n=k/2+1/2$ and $s=k/2-1/2$ (and, of course, $\varepsilon=0$). After some simplifications one does indeed get the right-hand side of (4.34) and Lemma 3 is finally proved.

Now we turn to the case that the side lengths a,b,c of the hexagon have the same parity and therefore to the evaluation of the determinant in Lemma 1 with $x=(a+b)/2$ and $y=(a+c)/2$ (see the beginning of this section).

Lemma 4. *Let a,b,c be integers and $a\equiv c \pmod 2$. Then the determinant*

$$-\det_{1\le i,j\le a+1}\;\begin{array}{c} \begin{array}{cc} \quad 1\le j\le a \quad & \quad j=a+1 \quad \end{array} \\ \left(\begin{array}{c|c} \dbinom{b+c}{c-i+j} & \dbinom{\frac{b+c}{2}}{\frac{c}{2}-i+\frac{a+2}{2}} \\ \hline \dbinom{\frac{b+c-2}{2}}{\frac{c}{2}-\frac{a+2}{2}+j} & 0 \end{array}\right) \end{array} \begin{array}{l} 1\le i\le a \\ \\ i=a+1 \end{array}$$

is equal to

$$\left(\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}\right)\frac{(1)_c}{(b+1)_{c+a-1}}\binom{\frac{b+c-2}{2}}{\frac{b-1}{2}}\binom{\frac{a+b+c-1}{2}}{\frac{b-1}{2}}2^{a-1}$$
$$\times\left(\left(\frac{c+1}{2}\right)_{(a-1)/2}\left(\frac{b+c+2}{2}\right)_{(a-1)/2}\frac{(\frac{1}{2})_{(a-1)/2}}{(1)_{(a-1)/2}}\right.$$
$$+\sum_{k=1}^{(a-1)/2}\left(\frac{c+2}{2}\right)_{k-1}\left(\frac{b+c}{2}\right)_{k}\left(\frac{c+2k+1}{2}\right)_{(a-2k+1)/2}$$
$$\left.\times\left(\frac{b+c+2k+2}{2}\right)_{(a-2k-1)/2}\frac{(\frac{1}{2})_{(a-2k-1)/2}}{(1)_{(a-2k-1)/2}}\right)$$

in case that a is odd, and

$$\left(\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}\right)\frac{(1)_c}{(b)_{c+a}}\binom{\frac{b+c-2}{2}}{\frac{b-2}{2}}\binom{\frac{a+b+c-2}{2}}{\frac{b-2}{2}}2^{a}$$
$$\times\left(\left(\frac{c+2}{2}\right)_{(a-2)/2}\left(\frac{b+c+2}{2}\right)_{a/2}\frac{(\frac{1}{2})_{a/2}}{(1)_{(a-2)/2}}\right.$$
$$+\sum_{k=1}^{a/2}\left(\frac{c+1}{2}\right)_{k}\left(\frac{b+c}{2}\right)_{k}\left(\frac{c+2k+2}{2}\right)_{(a-2k)/2}$$
$$\left.\times\left(\frac{b+c+2k+2}{2}\right)_{(a-2k)/2}\frac{(\frac{1}{2})_{(a-2k)/2}}{(1)_{(a-2k)/2}}\right)$$

in case that a is even.

As already mentioned at the beginning of this section the proofs of Lemma 3 and Lemma 4 are quite similar. Therefore we restrict ourselves to just pointing out the differences to the proof of Lemma 3, but we omit the details. The only major difference to Lemma 3 is that in the present case the irreducible polynomial $P_a(b,c)$ (see, e.g., (4.29)) has one 'exceptional evaluation' which has to be treated in a different way than the rest. By the way, this exceptional evaluation also causes the formulas for the number of rhombus tilings which contain the 'almost central' rhombus above the centre to be not as simple compared to the formulas for the number of rhombus tilings which contain the central rhombus.

Proof of Lemma 4.

Step 1: From our determinant to a determinant with polynomial entries.

We can straightforwardly adopt Step 1 from the proof of Lemma 3, where we have reduced the problem to the evaluation a polynomial determinant, since we have not even specialised the coordinates of the fixed rhombus (x,y) there. In the following $\hat{D}_a(b,c)$ denotes the matrix underlying the determinant in (4.6) evaluated for $x=(a+b)/2$ and $y=(a+c)/2$.

Again I was led to a conjecture concerning the determinant $\det(\hat{D}_a(b,c))$ by computing it for small values of a:

$$\det_{2\le i,j\le a}\left(\hat{D}_a(b,c)\right)=\left(\prod_{i=2}^{a-1}(1+b+c)_{i-1}\right)\left(\prod_{i=2}^{a}(i-1)!\right)$$
$$\left(\frac{b+1}{2}\right)^2_{(a-1)/2}\left(\frac{c+1}{2}\right)_{(a-1)/2}\left(\frac{2+b+c}{2}\right)_{(a-1)/2}2^{a-1}$$
$$\times\left(\left(\frac{c+1}{2}\right)_{(a-1)/2}\left(\frac{b+c+2}{2}\right)_{(a-1)/2}\frac{(\frac{1}{2})_{(a-1)/2}}{(1)_{(a-1)/2}}\right.$$
$$+\sum_{k=1}^{(a-1)/2}\left(\frac{c+2}{2}\right)_{k-1}\left(\frac{b+c}{2}\right)_{k}\left(\frac{c+2k+1}{2}\right)_{(a-2k+1)/2}$$
$$\left.\times\left(\frac{b+c+2k+2}{2}\right)_{(a-2k-1)/2}\frac{(\frac{1}{2})_{(a-2k-1)/2}}{(1)_{(a-2k-1)/2}}\right)\tag{4.39}$$

if a is odd, and

$$\det_{2\le i,j\le a}\left(\hat{D}_a(b,c)\right)=\left(\prod_{i=2}^{a-1}(1+b+c)_{i-1}\right)\left(\prod_{i=2}^{a}(i-1)!\right)$$
$$\left(\frac{b}{2}\right)_{a/2}\left(\frac{b+2}{2}\right)_{(a-2)/2}\left(\frac{c+2}{2}\right)_{(a-2)/2}\left(\frac{2+b+c}{2}\right)_{(a-2)/2}2^{a-1}$$
$$\times\left(\left(\frac{c+2}{2}\right)_{(a-2)/2}\left(\frac{b+c+2}{2}\right)_{a/2}\frac{(\frac{1}{2})_{a/2}}{(1)_{(a-2)/2}}\right.$$
$$+\sum_{k=1}^{a/2}\left(\frac{c+1}{2}\right)_{k}\left(\frac{b+c}{2}\right)_{k}\left(\frac{c+2k+2}{2}\right)_{(a-2k)/2}$$
$$\left.\times\left(\frac{b+c+2k+2}{2}\right)_{(a-2k)/2}\frac{(\frac{1}{2})_{(a-2k)/2}}{(1)_{(a-2k)/2}}\right)\tag{4.40}$$

if a is even. Just as in the proof of Lemma 3 the two assertions (4.39) and (4.40) are equivalent to the assertion in Lemma 4.

The linear factors of $\det(\hat{D}_a(b,c))$ are

$$\prod_{i=2}^{a-1}(1+b+c)_{i-1}\left(\frac{b+1}{2}\right)^2_{(a-1)/2}\left(\frac{c+1}{2}\right)_{(a-1)/2}\left(\frac{2+b+c}{2}\right)_{(a-1)/2}2^{2a-2}\tag{4.41}$$

if a is odd, and, respectively,

$$\prod_{i=2}^{a-1}(1+b+c)_{i-1}\left(\frac{b}{2}\right)_{a/2}\left(\frac{b+2}{2}\right)_{(a-2)/2}\left(\frac{c+2}{2}\right)_{(a-2)/2}\left(\frac{2+b+c}{2}\right)_{(a-2)/2}2^{2a-3}\tag{4.42}$$

if a is even. The irreducible polynomial, denoted by $\hat{P}_a(b,c)$, reads

$$\left(\prod_{i=2}^{a}(i-1)!\right)\left(\frac{1}{2}\right)^{a-1}\left(\left(\frac{c+1}{2}\right)_{(a-1)/2}\left(\frac{b+c+2}{2}\right)_{(a-1)/2}\frac{\left(\frac{1}{2}\right)_{(a-1)/2}}{(1)_{(a-1)/2}}\right.$$
$$+\sum_{k=1}^{(a-1)/2}\left(\frac{c+2}{2}\right)_{k-1}\left(\frac{b+c}{2}\right)_{k}\left(\frac{c+2k+1}{2}\right)_{(a-2k+1)/2}$$
$$\left.\times\left(\frac{b+c+2k+2}{2}\right)_{(a-2k-1)/2}\frac{\left(\frac{1}{2}\right)_{(a-2k-1)/2}}{(1)_{(a-2k-1)/2}}\right) \tag{4.43}$$

if a is odd, and respectively,

$$\left(\prod_{i=2}^{a}(i-1)!\right)\left(\frac{1}{2}\right)^{a-2}\left(\left(\frac{c+2}{2}\right)_{(a-2)/2}\left(\frac{b+c+2}{2}\right)_{a/2}\frac{\left(\frac{1}{2}\right)_{a/2}}{(1)_{(a-2)/2}}\right.$$
$$+\sum_{k=1}^{a/2}\left(\frac{c+1}{2}\right)_{k}\left(\frac{b+c}{2}\right)_{k}\left(\frac{c+2k+2}{2}\right)_{(a-2k)/2}$$
$$\left.\times\left(\frac{b+c+2k+2}{2}\right)_{(a-2k)/2}\frac{\left(\frac{1}{2}\right)_{(a-2k)/2}}{(1)_{(a-2k)/2}}\right) \tag{4.44}$$

if a is even.

Step 2: The linear factors in (4.41), respectively in (4.42), divide $\det(\hat{D}_a(b,c))$ as a polynomial in b and c.

Again we have four different types of linear factors:

1. Factors of the form $(c+k)$: If a is odd, $k=1,3,\dots,a-2$. Otherwise $k=2,4,\dots,a-2$.
2. Factors of the form $(b+k)^2$: If a is odd, $k=1,3,\dots,a-2$. Otherwise $k=2,4,\dots,a-2$. The factor b occurs once if a even.
3. Factors of the form $(b+c+k)^{a-k-1}$ for $k=1,3,\dots,2\lceil(a-2)/2\rceil-1$.
4. Factors of the form $(b+c+k)^{a-k}$ for $k=2,4,\dots,2\lfloor(a-2)/2\rfloor$.

re 1. — The factors of the form $(c+k)$ divide $\det(\hat{D}_a(b,c))$: As described in the analogous passage of the proof of Lemma 3, each linear factor of $\det(\hat{D}_a(b,c))$ corresponds to a linear combination of rows of a certain matrix. The linear combinations for the factors of the form $(c+k)$ are

$$\sum_{i=(a-k+2)/2}^{a-k+1}\frac{(-1)^{i-1}(b+i)_{a-k+1-i}\left(\frac{-a+k-2+2i}{2}\right)_{a-k+1-i}}{(1)_{a-k+1-i}\left(\frac{b-a+2i-2}{2}\right)_{a-k+1-i}}D_a(b,-k)_{(i,j)}=0 \tag{4.45}$$

for $j=2,3,\dots,a$, $1\le k\le a-1$ and $k\equiv a \pmod 2$. In order to prove that $(c+k)$ divides the determinant $\det(\hat{D}_a(b,c))$ we just have to prove the identity (4.45).

The comparison of the identity in (4.13) and the identity in (4.45) shows that these two identities are quite similar. And in fact the proofs do not differ essentially from each other, either: Again we are able to apply *Gosper's algorithm* [7] for hypergeometric sums to the left-hand side of (4.45) and recognize that this is actually a telescoping sum.

The expression analogous to (4.14) is

$$\sum_{i=0}^{\infty}\frac{(-1)^j\left(1-\frac{a}{2}+\frac{k}{2}\right)_i(-a-b+k)_{i+j-2}}{(1)_{-a+i+j}\left(1-\frac{a}{2}-\frac{b}{2}+k\right)_i}\,H|_{x=(a+b)/2,y=(a+c)/2,c=-k,i\to a-k+1-i}=0, \tag{4.46}$$

which is the identity in (4.45) after reversing the summation order. If $\hat{f}(i)$ denotes the summand in the previous sum and $\hat{g}(i)$ denotes the following expression

$$\frac{(-1)^j \left(1-\frac{a}{2}+\frac{k}{2}\right)_i (-a-b+k)_{i+j-2}}{(1)_{-1-a+i+j} \left(1-\frac{a}{2}-\frac{b}{2}+k\right)_{i-1}} \times \left(\frac{-2-a-b-bi+2j+bj-bk+ik-jk+k^2}{2}\right),$$

then we have

$$\hat{f}(i) = \hat{g}(i+1) - \hat{g}(i).$$

By using this identity it is easy to compute the left-hand side of (4.46) and to show that it is equal to zero.

re 2. — The factors of the form $(b+k)$ *divide* $\det(\hat{D}_a(b,c))$*:*

The two linearly independent linear combination for the factors of the form are $(b+k)$ are

$$\sum_{i=k+2}^{(a+k+2)/2} \frac{(-1)^{i-k-2}(c+a-i+2)_{i-k} \left(\frac{a+k-2i+4}{2}\right)_{i-k-2}}{(1)_{i-k-1} \left(\frac{c+a-2i+4}{2}\right)^2_{i-k-2}} \times p_{i-k-1}(c+a-k-i+2)\hat{D}_a(-k,c)_{(i,j)} = 0 \quad (4.47)$$

if $0 \le k < a-2$ and $k \equiv a \pmod 2$, and

$$\sum_{i=k+3}^{(a+k+2)/2} \frac{(-1)^{i-k}(c+a-i+2)_{i-k-1} \left(\frac{a+k-2i+4}{2}\right)_{i-k-1}}{(1)_{i-k-1} \left(\frac{c+a-2i+4}{2}\right)^2_{i-k-2}} \times p_{i-k-2}(c+a-k-i+1)\hat{D}_a(-k,c)_{(i,j)} = -\hat{D}_a(-k,c)_{(k+1,j)} \quad (4.48)$$

if $0 < k < a-2$ and $k \equiv a \pmod 2$, where $p_n(c)$ is the same sequence of polynomials as in Lemma 3, see (4.17). The first identity, (4.47), corresponds to (4.15), and the second identity, (4.48), corresponds to (4.16). Again these two linear combinations do not cover the case that $k = a-2$. But similar to the situation in Lemma 3 it can be verified directly that $(b+a-2)^2$ is a factor of the determinant $\det(\hat{D}_a(b,c))$.

The proofs of the identities (4.47) and (4.48) are analogous to the proofs of their corresponding identity: Again we first have to interchange the summation and then split the double sum into four smaller sums according to the polynomial H. Just as in Lemma 3 the inner sums of the first and the third double sum nearly cancel out each other, and so do the inner sums of the second and the fourth summand. This reduces the problem to identities only involving ${}_2F_1$–series and therefore Vandermonde's summation formula (4.22) finishes the proof of these identities.

re 3., 4. — $\prod_{i=2}^{a}(b+c+1)_{i-1}/((b+c+1)/2)_{\lceil (a-1)/2 \rceil}$ *is a factor of* $\det(\hat{D}_a(b,c))$*:* Just as in Lemma 3 the triple sum from Lemma 2 provides an easy proof that the factors of type 3 and type 4 divide the determinant $\det(\hat{D}_a(b,c))$ as a polynomial in b and c. Again a combination of (4.3), (4.5) and Lemma 2 gives the equation analogous to (4.26), namely

$$\det(\hat{D}_a(b,c)) = \left(\prod_{i=2}^{a}(b+c+1)_{i-1}(i-1)!\right)\left(\frac{c-a+2}{2}\right)_{a-1}\left(\frac{b-a+2}{2}\right)_{a-1}$$
$$\times\sum_{n=1}^{a}\sum_{m=1}^{a}\sum_{s=1}^{m}\left[(-1)^{n+s}\frac{\left(\frac{-a+c+2n}{2}\right)_{(a+b-2)/2}}{\left(\frac{2-a+c}{2}\right)_{(a+b-2)/2}}\frac{\left(\frac{-a+b+2s}{2}\right)_{(a+c)/2}}{\left(\frac{2-a+b}{2}\right)_{(a+c)/2}}\binom{m-1}{s-1}\right.$$
$$\left.\frac{(b+1)_{s-1}}{(b+c+1)_{s-1}}\frac{(c+1)_{n-1}}{(n-1)!}\frac{(b+c+n)_{m-n}}{(m-n)!}\right]. \tag{4.49}$$

By using (4.28) we obtain that the quotient of (4.49) and the product of the linear factors $\prod_{i=2}^{a}(b+c+1)_{i-1}/((b+c+1)/2)_{\lceil(a-1)/2\rceil}$ is the following polynomial in b and c if we fix a:

$$\prod_{i=2}^{a}(i-1)!\sum_{n=1}^{a}\sum_{m=1}^{a}\sum_{s=1}^{m}(-1)^{n+s}\left(\frac{-a+c+2n}{2}\right)_{a-n}\left(\frac{b+c}{2}\right)_{n-1}\left(\frac{-a+b+2s}{2}\right)_{a-s}$$
$$\times\binom{m-1}{s-1}\frac{\left(\frac{b+c+1}{2}\right)_{\lceil(a-1)/2\rceil}\left(\frac{b+c+2}{2}\right)_{s-1}}{(b+c+1)_{s-1}}\frac{(b+1)_{s-1}(c+1)_{n-1}(b+c+n)_{m-n}}{(n-1)!(m-n)!}.$$

Step 3: Computation of the irreducible polynomial. Analogously to the situation in Lemma 2 we now evaluate the quotient of the linear factors in (4.41), respectively (4.42), and the determinant $\det(\hat{D}_a(b,c))$. This quotient, which will be denoted by $\hat{P}_a(b,c)$, is a polynomial in b and c if we fix a. Again we will find that the polynomial $\hat{P}_a(b,c)$ has enough 'nice' evaluations so that we can compute the polynomial by using Lagrange's interpolation formula. Namely, these 'nice' evaluations arise for $b=-c-k$, $0\le k\le a$ and k is even, and the degree of $\hat{P}_a(b,c)$ as a polynomial in b happens to be $\lfloor a/2\rfloor$.

The assertion about the degree of $\hat{P}_a(b,c)$ as a polynomial in b can be checked routinely, just as in Lemma 3. Concerning the computation of 'nice' evaluations of the polynomial $\hat{P}_a(b,c)$ there arises the one and only major difference to Lemma 3: For $k=2,4,\dots,\lfloor a/2\rfloor$ the evaluations $\hat{P}_a(-c-k,c)$ can be proved in the same way as in Lemma 3, but for $k=0$ the situation is different, and therefore this exceptional evaluation needs a separate proof. This fact already shows up if we look at the following conjecture for the 'nice' evaluations:

We claim that

$$\hat{P}_a(-c,c) = \left(\frac{1}{2}\right)^{a-1}\left(\prod_{i=2}^{a}(i-1)!\right)\left(\frac{1}{2}\right)_{(a-1)/2}\left(\frac{c+1}{2}\right)_{(a-1)/2} \tag{4.50}$$

if a is odd, and

$$\hat{P}_a(-c,c) = \left(\frac{1}{2}\right)^{a-2}\left(\prod_{i=2}^{a}(i-1)!\right)\left(\frac{a}{2}\right)\left(\frac{1}{2}\right)_{a/2}\left(\frac{c+2}{2}\right)_{(a-2)/2} \tag{4.51}$$

if a is even, and, furthermore,

$$\hat{P}_a(-c-k,c) = \left(\frac{1}{2}\right)^{a-1}\left(\prod_{i=2}^{a}(i-1)!\right)\left(\frac{1}{2}\right)_{(a-1)/2}\frac{\left(\frac{k}{2}\right)!(-1)^{k/2}}{\left(\frac{a-k}{2}\right)_{k/2}}$$
$$\times\left(\frac{c+k+1}{2}\right)_{(a-k+1)/2}\left(\frac{c+2}{2}\right)_{(k-2)/2} \tag{4.52}$$

if a is odd and $k=2,4,\ldots,a-1$, and

$$\hat{P}_a(-c-k,c)=\left(\frac{1}{2}\right)^{a-2}\prod_{i=1}^{a-1} i!\left(\frac{1}{2}\right)_{(a-2)/2}\frac{\left(\frac{k}{2}\right)!(-1)^{k/2}}{\left(\frac{a-k+1}{2}\right)_{(k-2)/2}} \times\left(\frac{c+k+2}{2}\right)_{(a-k)/2}\left(\frac{c+1}{2}\right)_{k/2} \tag{4.53}$$

if a is even and $k=2,4,\ldots,a$.

Assuming the truth of the claim, we would have

$$\hat{P}_a(b,c)=\left(\frac{1}{2}\right)^{a-1}\left(\prod_{i=2}^{a}(i-1)!\right)\Bigg(\left(\frac{c+1}{2}\right)_{(a-1)/2}\left(\frac{b+c+2}{2}\right)_{(a-1)/2}\frac{\left(\frac{1}{2}\right)_{(a-1)/2}}{(1)_{(a-1)/2}} + \sum_{k=1}^{(a-1)/2}\left(\frac{c+2}{2}\right)_{k-1}\left(\frac{b+c}{2}\right)_{k}\left(\frac{c+2k+1}{2}\right)_{(a-2k+1)/2} \times\left(\frac{b+c+2k+2}{2}\right)_{(a-2k-1)/2}\frac{\left(\frac{1}{2}\right)_{(a-2k-1)/2}}{(1)_{(a-2k-1)/2}}\Bigg)$$

if a is odd, and, respectively

$$\hat{P}_a(b,c)=\left(\frac{1}{2}\right)^{a-2}\prod_{i=2}^{a}(i-1)!\Bigg(\left(\frac{c+2}{2}\right)_{(a-2)/2}\left(\frac{b+c+2}{2}\right)_{a/2}\frac{\left(\frac{1}{2}\right)_{a/2}}{(1)_{(a-2)/2}} + \sum_{k=1}^{a/2}\left(\frac{c+1}{2}\right)_{k}\left(\frac{b+c}{2}\right)_{k}\left(\frac{c+2k+2}{2}\right)_{(a-2k)/2} \times\left(\frac{b+c+2k+2}{2}\right)_{(a-2k)/2}\frac{\left(\frac{1}{2}\right)_{(a-2k)/2}}{(1)_{(a-2k)/2}}\Bigg)$$

if a is even, by Lagrange interpolation.

Thus, it remains to show (4.50), (4.51), (4.53) and (4.52). In these claims we replace the polynomial $\hat{P}_a(b,c)$ by the quotient of $\det(\hat{D}_a(b,c))$ and the linear factors in (4.41), respectively (4.42), in order to see that the four claims conflate to the following two:

$$\left.\frac{2\det(\hat{D}_a(b,c))\binom{\frac{b+c-2}{2}}{\frac{a+b-2}{2}}\binom{\frac{b+c}{2}}{\frac{a+c}{2}}}{\left(\prod_{i=2}^{a}(1+b+c)_{i-1}(i-1)!\right)\left(\frac{c-a+2}{2}\right)_{a-1}\left(\frac{b-a+2}{2}\right)_{a-1}(b+c)}\right|_{b=-c} =\frac{-a+(-1)^{a+1}\,a+c+(-1)^{a+1}\,c}{c\,(a+c)}, \tag{4.54}$$

and

$$\left.\frac{\det(\hat{D}_a(b,c))\binom{\frac{b+c-2}{2}}{\frac{a+b-2}{2}}\binom{\frac{b+c}{2}}{\frac{a+c}{2}}}{\left(\prod_{i=2}^{a}(1+b+c)_{i-1}(i-1)!\right)\left(\frac{c-a+2}{2}\right)_{a-1}\left(\frac{b-a+2}{2}\right)_{a-1}}\right|_{b=-c-k} = (-1)^{a-1}\frac{(c+1)_{k-1}}{(k-1)!} \tag{4.55}$$

for $k = 2, 4, \dots, \lfloor a/2 \rfloor$. Again the reader should know that we are not able to directly set $b = -c-k$ on the left hand sides of (4.55) and (4.54), because the denominator vanishes for $b = -c-k$.

First we consider the case that $k \neq 0$. As already mentioned, the situation in this case is quite the same as in Lemma 3. Therefore I only explain the essential steps and omit details.

For the left-hand side of (4.55) we use again the modification of Lemma 2, see (4.35) and (4.36). Thus, this left-hand side is equal to

$$\lim_{\varepsilon\to 0}\sum_{n=1}^{a}\sum_{m=1}^{a}\sum_{s=1}^{m}\Bigg[(-1)^{n+s}\binom{\frac{-k+2n-4}{2}}{\frac{a-c-k-2}{2}}\binom{\frac{-k+2s-2}{2}}{\frac{a+c}{2}}\binom{m-1}{s-1} \\ \times\frac{(-c-k+\varepsilon+1)_{s-1}}{(-k+\varepsilon+1)_{s-1}}\frac{(c+1)_{n-1}}{(n-1)!}\frac{(-k+\varepsilon+n)_{m-n}}{(m-n)!}\Bigg]. \tag{4.56}$$

Hence, it remains to show that the right hand side of (4.55) is equal to (4.56) for $k = 2, 4, \dots, \lfloor a/2 \rfloor$.

Just as in Lemma 2 we apply the transformation formula due to Thomae, see (4.38), to the innermost sum of the triple sum, with slightly changed parameters, i.e., $a = 1-m$, $b = 1-c-k+\varepsilon$, $n = (k-2)/2$, $d = 1-k+\varepsilon$ and $e = 1-a/2-c/2-k/2$, compared to the proof of Lemma 2. This yields

$$\lim_{\varepsilon\to 0}\sum_{n=1}^{a}\sum_{m=1}^{a}\sum_{s=0}^{\infty}(-1)^{\frac{a}{2}+\frac{c}{2}+n+s}\left(\frac{-a}{2}+\frac{c}{2}+\frac{k}{2}\right) \\ \times\frac{(1+c)_{-1+n}\,(1-c-k)_s\,(\frac{-a}{2}+\frac{c}{2}+n)_{-1+\frac{a}{2}-\frac{c}{2}-\frac{k}{2}}\,(\varepsilon-k+n)_{m-n+s}}{(1)_{m-n}\,(1)_{-1+n}\,(1)_{-1+\frac{k}{2}-s}\,(1)_s\,(1)_{1+\frac{a}{2}-\frac{c}{2}-k+s}\,(1+\varepsilon-k)_s}.$$

Next we interchange the two inner sums, reverse the order of the summation in the new innermost sum and use hypergeometric notation for this innermost sum. We obtain

$$\lim_{\varepsilon\to 0}\sum_{n=1}^{m}\sum_{s=0}^{\infty}(-1)^{\frac{a}{2}+\frac{c}{2}-n+s}\,{}_2F_1\left[\begin{matrix}1,-a+n\\1-a-\varepsilon+k-s\end{matrix};1\right] \\ \times\frac{(1+c)_{-1+n}\,(1-c-k)_s\,(\frac{-a}{2}+\frac{c}{2}+\frac{k}{2})\,(\frac{-a}{2}+\frac{c}{2}+n)_{-1+\frac{a}{2}-\frac{c}{2}-\frac{k}{2}}\,(\varepsilon-k+n)_{a-n+s}}{(1)_{a-n}\,(1)_{-1+n}\,(1)_{-1+\frac{k}{2}-s}\,(1)_s\,(1)_{1+\frac{a}{2}-\frac{c}{2}-k+s}\,(1+\varepsilon-k)_s}.$$

Then we apply Vandermonde's summation formula (4.22) to this hypergeometric sum. After some manipulations this gives

$$\lim_{\varepsilon\to 0}\sum_{n=1}^{a}\sum_{s=0}^{\infty}(-1)^{-1-\frac{k}{2}}\left(\frac{-a}{2}+\frac{c}{2}+\frac{k}{2}\right)(1+c)_{-1+n}\,(1-c-k)_s \\ \times\frac{(-a-\varepsilon+k-s)_{a-n}\,(1-\varepsilon+k-n-s)_s\,(2+\frac{a}{2}-\frac{c}{2}-k+s)_{-1+k-n-s}}{(1)_{a-n}\,(1)_{1+\frac{k}{2}-n}\,(1)_{-1+n}\,(1)_{-1+\frac{k}{2}-s}\,(1)_s\,(1+\varepsilon-k)_s}.$$

Now we are able to perform the limit $\varepsilon\to 0$ and then observe that the summand of the double sum is only different from zero if $n=k/2+1$ and $s=k/2-1$. The evaluation of the summand of the double sum for these special values of n and s finishes the proof of (4.55) for $k=2,4,\ldots\lfloor a/2\rfloor$.

Finally we consider the case that $k=0$, see (4.54). By Lemma 2, (4.3) and (4.5) the left-hand side of (4.54) is equal to

$$\frac{2}{b+c}\sum_{n=1}^{a}\sum_{m=1}^{a}\sum_{s=1}^{m}\frac{(-1)^{n+s}\,(1+b)_{-1+s}\,(1+c)_{-1+n}}{(1)_{-1+s}\,(1)_{-1+n}} \\ \times\frac{(1+\frac{a}{2}+\frac{c}{2})_{-1-\frac{a}{2}+\frac{b}{2}+s}\,(\frac{-a}{2}+\frac{c}{2}+n)_{-1+\frac{a}{2}+\frac{b}{2}}\,(b+c+n)_{m-n}\,(1+m-s)_{-1+s}}{(1)_{-1+\frac{a}{2}+\frac{b}{2}}\,(1)_{m-n}\,(1)_{-1-\frac{a}{2}+\frac{b}{2}+s}\,(1+b+c)_{-1+s}}.$$

before evaluating it at $b=-c$. We take the factor $(b+c)/2$ out of the Pochhammer symbol

$$\left(\frac{2+a+c}{2}\right)_{(-2-a+b+2s)/2}=\left(\frac{2+a+c}{2}\right)_{(-2+a+b)/2}\left(\frac{b}{2}+\frac{c}{2}\right)\left(\frac{b+c+2}{2}\right)_{s-1}$$

and set $b=-c$. After some cancellations and simplifications concerning the summand of this triple sum, this yields

$$\sum_{n=1}^{a}\sum_{m=1}^{a}\sum_{s=1}^{m}\frac{(-1)^{-1+\frac{a}{2}-\frac{c}{2}+n+s}\,(1-c)_{-1+s}\,(1+c)_{-1+n}}{(1)_{-1+n}\,(1)_{-1+s}} \\ \times\frac{(\frac{a}{2}-\frac{c}{2})_{1-n}\,(1+\frac{a}{2}+\frac{c}{2})_{-1-\frac{a}{2}-\frac{c}{2}}\,(n)_{m-n}\,(1+m-s)_{-1+s}}{(1)_{1-n}\,(1)_{m-n}\,(1)_{-1-\frac{a}{2}-\frac{c}{2}+s}}. \tag{4.57}$$

The factor $1/(1)_{1-n}$ shows that the summand of the triple sum is only different from zero if $n=1$. Therefore we get rid of the sum over n. We use hypergeometric notation for the inner sum and obtain

$$\sum_{m=1}^{a}\frac{(-1)^{1+c}\,{}_2F_1\left[\begin{matrix}1-c,1-m\\1-\frac{a}{2}-\frac{c}{2}\end{matrix};1\right]}{\frac{a}{2}+\frac{c}{2}}.$$

Next we apply Vandermonde's summation formula (4.22) to the ${}_2F_1$–series. This yields

$$\sum_{m=1}^{a}\frac{(-1)^{c}\,(\frac{-a}{2}+\frac{c}{2})_{-1+m}}{(\frac{-a}{2}-\frac{c}{2})_{m}}. \tag{4.58}$$

Once more, Gosper's algorithm [7] can be used to see that this single sum is actually telescoping. Namely, there holds

$$(-1)^c \frac{\left(-\frac{a}{2}+\frac{c}{2}\right)_{-1+m}}{\left(-\frac{a}{2}-\frac{c}{2}\right)_m}$$
$$= \frac{(-1)^{c-1}}{\left(1+\frac{a}{2}+\frac{c}{2}\right)c}\left(\frac{\left(-\frac{a}{2}-\frac{c}{2}+m\right)\left(-1-\frac{a}{2}+\frac{c}{2}\right)_{m+1}}{\left(-\frac{a}{2}-\frac{c}{2}\right)_{m+1}} - \frac{\left(-\frac{a}{2}-\frac{c}{2}+m-1\right)\left(-1-\frac{a}{2}+\frac{c}{2}\right)_m}{\left(-\frac{a}{2}-\frac{c}{2}\right)_m}\right).$$

Because of that, and because a and c have the same parity, the single sum (4.58) is equal to

$$\frac{-a+(-1)^{a+1}\,a+c+(-1)^{a+1}\,c}{c\,(a+c)}.$$

If we compare this to the right-hand side of (4.54) we see that this is just what we claimed. This finally completes the proof of Lemma 4.

5. The Proofs of Theorem 3 and Theorem 4

We end this article with the proofs of Theorem 3 and Theorem 4. We start with Theorem 3. In Theorem 1 we showed that the probability to choose a rhombus tilings of a hexagon with side lengths a,b,c,a,b,c which contains a rhombus in the centre is, using hypergeometric notation,

$$ {}_4F_3\left[\begin{matrix}\frac{1}{2}-\frac{a}{2},\frac{1}{2}+\frac{c}{2},\frac{1}{2}+\frac{b}{2}+\frac{c}{2},1\\ 1-\frac{a}{2},1+\frac{c}{2},\frac{3}{2}+\frac{b}{2}+\frac{c}{2}\end{matrix};1\right]$$
$$\times\frac{\left(\frac{1}{2}\right)_{-\frac{1}{2}+\frac{a}{2}}(1)_c\left(1+\frac{c}{2}\right)_{-\frac{1}{2}+\frac{b}{2}}\left(1+\frac{c}{2}\right)_{-1+\frac{a}{2}+\frac{b}{2}}\left(\frac{3}{2}+\frac{b}{2}+\frac{c}{2}\right)_{-\frac{1}{2}+\frac{a}{2}}2^{a-1}}{(1)_{-\frac{1}{2}+\frac{a}{2}}\,{(1)_{-\frac{1}{2}+\frac{b}{2}}}^2\,(1+b)_{-1+a+c}} \tag{5.1}$$

in case that a is odd, and

$$ {}_4F_3\left[\begin{matrix}1-\frac{a}{2},1+\frac{c}{2},\frac{1}{2}+\frac{b}{2}+\frac{c}{2},1\\ \frac{3}{2}-\frac{a}{2},\frac{3}{2}+\frac{c}{2},\frac{3}{2}+\frac{b}{2}+\frac{c}{2}\end{matrix};1\right]$$
$$\times\frac{\left(\frac{1}{2}\right)_{-1+\frac{a}{2}}(1)_c\,(b)\left(\frac{1}{2}+\frac{c}{2}\right)_{\frac{b}{2}}\left(\frac{3}{2}+\frac{c}{2}\right)_{-1+\frac{a}{2}+\frac{b}{2}}\left(\frac{3}{2}+\frac{b}{2}+\frac{c}{2}\right)_{-1+\frac{a}{2}}2^{a-2}}{(1)_{-1+\frac{a}{2}}\,{(1)_{\frac{b}{2}}}^2\,(1+b)_{-1+a+c}} \tag{5.2}$$

in case that a is even. In order to see this, divide (1.2), respectively (1.3), by MacMahon's formula for the total number of rhombus tilings of a hexagon with side lengths a,b,c,a,b,c given in (1.1).

Next we apply Bailey's transformation formula (see [13, (4.3.5.1)]) between two balanced ${}_4F_3$–series,

$$ {}_4F_3\left[\begin{matrix}a,b,c,-n\\ e,f,1+a+b+c-e-f-n\end{matrix};1\right]=\frac{(e-a)_n(f-a)_n}{(e)_n(f)_n}\times$$
$${}_4F_3\left[\begin{matrix}-n,a,1+a+c-e-f-n,1+a+b-e-f-n\\ 1+a+b+c-e-f-n,1+a-e-n,1+a-f-n\end{matrix};1\right], \tag{5.3}$$

which is valid if n is a positive integer, with $a=1, b=(1+c)/2, c=(1+b+c)/2, n=(a-1)/2, e=(2-a)/2, f=(c+2)/2$ in case of a is odd, and with $a=1, b=(2+c)/2, c=$

$(1+b+c)/2, n=(a-2)/2, e=(3-a)/2, f=(c+3)/2$ in case of a is even. This gives

$$
{}_4F_3\left[\begin{matrix}1,1,\frac{1}{2}-\frac{a}{2},1+\frac{b}{2}\\ \frac{3}{2},\frac{3}{2}+\frac{b}{2}+\frac{c}{2},\frac{3}{2}-\frac{a}{2}-\frac{c}{2}\end{matrix};1\right]
\times\frac{a!b!c!((a+b+c-2)/2)!((a+b+c)/2)!}{((a-1)/2)!^2((b-1)/2)!^2(c/2)!((c-2)/2)!(a+b+c-1)!((-1+a+c)/2)((1+b+c)/2)}
$$

if a is odd, and

$$
{}_4F_3\left[\begin{matrix}1,1,1-\frac{a}{2},\frac{1}{2}+\frac{b}{2}\\ \frac{3}{2},\frac{3}{2}+\frac{b}{2}+\frac{c}{2},\frac{3}{2}-\frac{a}{2}-\frac{c}{2}\end{matrix};1\right]
\times\frac{(a-1)!b!c!((a+b+c-1)/2)!^2}{((a-2)/2)!^2(b/2)!^2((c-1)/2)!^2(a+b+c-1)!((-1+a+c)/2)((1+b+c)/2)}
$$

if a is even, after transforming the Pochhammer symbol $(a)_n=(a+n-1)!/(a-1)!$ to factorials.

Now we substitute $a\sim\alpha N$, $b\sim\beta N$ and $c\sim\gamma N$ and perform the limit $N\to\infty$. We use Stirling's formula to determine the limit for the quotient of the factorials in the second line as $\sqrt{\alpha}\sqrt{\beta}\sqrt{\gamma}\sqrt{\alpha+\beta+\gamma}/(2\pi(\alpha+\gamma)(\beta+\gamma))$. For the ${}_4F_3$–series, we may exchange limit and summation by uniform convergence:

$$
\lim_{N\to\infty}{}_4F_3\left[\begin{matrix}1,1,\frac{1}{2}-\frac{a}{2},1+\frac{b}{2}\\ \frac{3}{2},\frac{3}{2}+\frac{b}{2}+\frac{c}{2},\frac{3}{2}-\frac{a}{2}-\frac{c}{2}\end{matrix};1\right]={}_2F_1\left[\begin{matrix}1,1\\ \frac{3}{2}\end{matrix};\frac{\alpha\beta}{(\beta+\gamma)(\alpha+\gamma)}\right].
$$

A combination of these results and the use of the identity

$$
{}_2F_1\left[\begin{matrix}1,1\\ \frac{3}{2}\end{matrix};z\right]=\frac{\arcsin\sqrt{z}}{\sqrt{z(1-z)}}
$$

establishes Theorem 3.

The proof of Theorem 4 is analogous: Again we divide the formulas in Theorem 2, see (1.4) and (1.3), by MacMahon's formula for the total number of rhombus tilings of a hexagon with side lengths a, b, c, a, b, c given in (1.1), and obtain the probability to choose a rhombus tiling of a hexagon with side lengths a, b, c, a, b, c which contain the 'almost central' rhombus above the centre. Next we apply Bailey's transformation formula (5.3) on the hypergeometric sum and transform the Pochhammer symbols to factorials. We then substitute $a\sim\alpha N$, $b\sim\beta N$ and $c\sim\gamma N$ and perform the limit $N\to\infty$ in the same way as before. The additional summands (caused by the 'exceptional' evaluation of the irreducible polynomial $\hat{P}_a(b,c)$) that appear in the formulas in Theorem 2 (when compared to the formulas in Theorem 1) vanish when this limit is performed.

References

[1] M. Ciucu and C. Krattenthaler, The number of centered lozenge tilings of a symmetric hexagon, J. Combin. Theory Ser. A 86 (1999), 103 - 126.

[2] H. Cohen, M. Larsen and J. Propp, The shape of a typical boxed plane partition, New York J. Math. 4 (1998), 137-166.

[3] G. David and C. Tomei, The problem of the calissons, Amer. Math. Monthly 96 (1989), 429-431.

[4] M. Fulmek and C. Krattenthaler, The number of rhombus tilings of a symmetric hexagon which contain a fixed rhombus on the symmetry axis, II, to appear.

[5] G. Gasper and M. Rahman, Basic hypergeometric series, Encyclopedia of mathematics and its applications 35, Cambridge University Press, Cambridge, 1990.

[6] I. M. Gessel and G. Viennot, Determinants, paths and plane partitions, preprint, 1989.

[7] R. W. Gosper, Decision procedure for indefinite hypergeometric summation, Proc. Natl. Acad. Sci. USA 75 (1978), 40-42.

[8] C. Krattenthaler, HYP and HYPQ - Mathematica packages for the manipulation of binomial sums and hypergeometric series, respectively q-binomial sums and basic hypergeometric series, J. Symbol. Comput. 20 (1995), 737-744.
The *Mathematica* package HYP is available via Internet at
`http://radon.mat.univie.ac.at/People/kratt/hyp_hypq/hyp.html`.

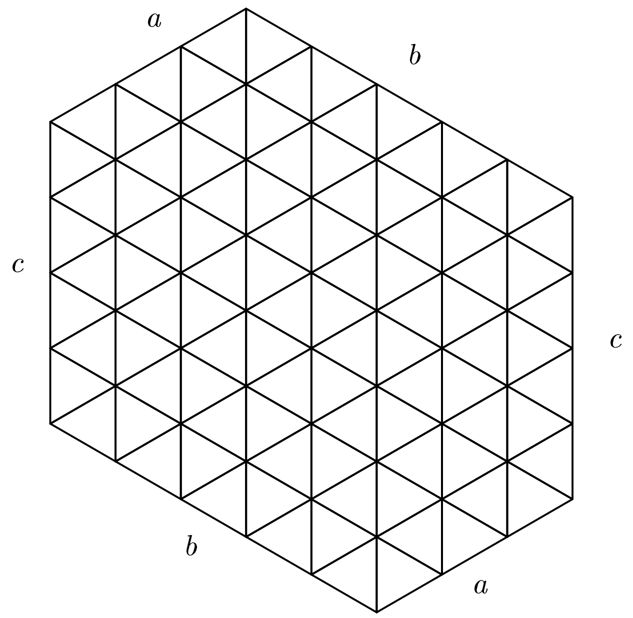
[9] C. Krattenthaler, Advanced determinant calculus, Séminaire Lotharingien Combin. 42 (1999), to appear.

[10] B. Lindström, On the vector representation of induced matroids, Bull. London Math. Soc. 5 (1973), 85-90.

[11] P. A. MacMahon, Combinatory Analysis, Vol. 2, Cambridge University Press, 1916; reprinted by Chelsea, New York, 1960.

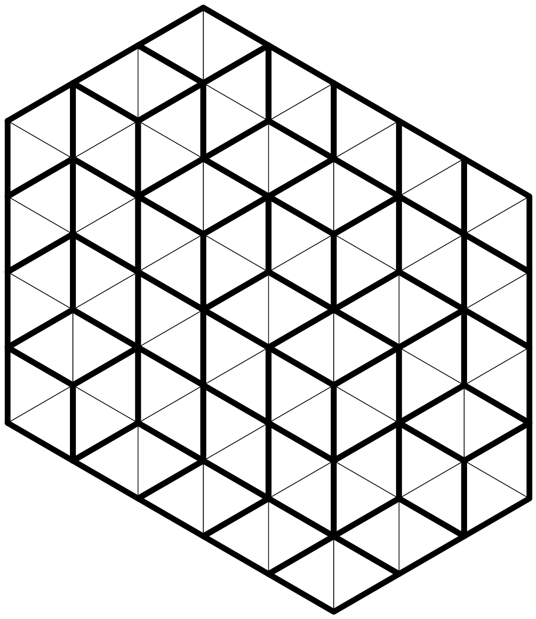
[12] P. Paule and M. Schorn, A Mathematica version of Zeilberger's algorithm for proving binomial identities, J. Symbol. Comp. 20 (1995), 673-698.

[13] L. J. Slater, Generalized hypergeometric functions, Cambridge University Press, Cambridge, 1966.



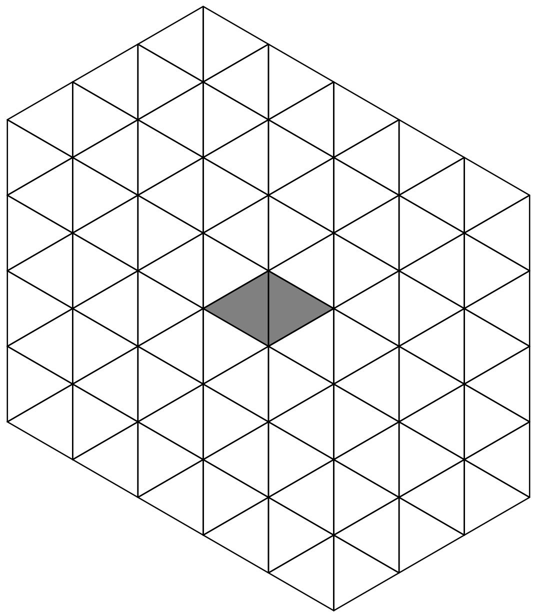
A hexagon with side lengths a, b, c, a, b, c, where $a = 3$, $b = 5$, $c = 4$.

FIGURE 1



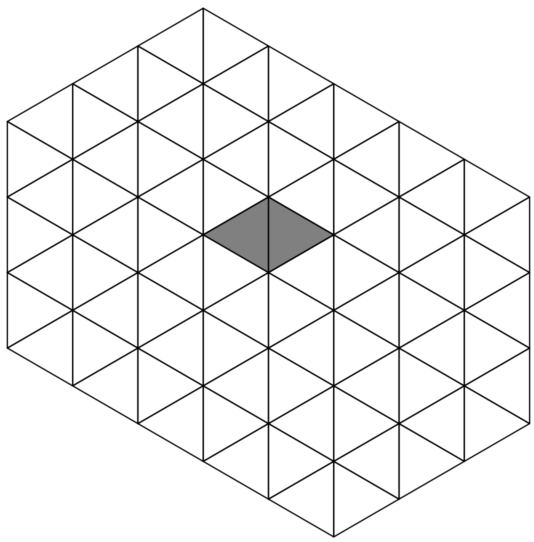
A rhombus tiling of a hexagon with side lengths a, b, c, a, b, c.

FIGURE 2



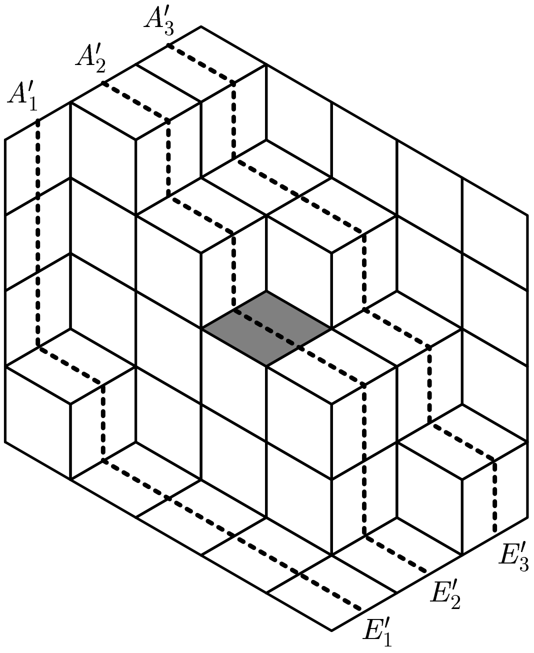
A hexagon with side lengths a, b, c, a, b, c, where the central rhombus is marked.

Figure 3



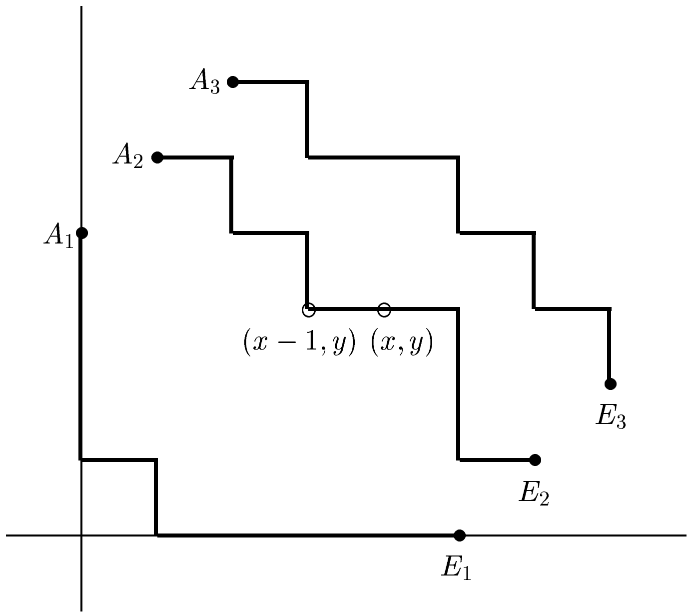
A hexagon with side lengths a, b, c, a, b, c,
where the 'almost central' rhombus above the center is marked.

Figure 4



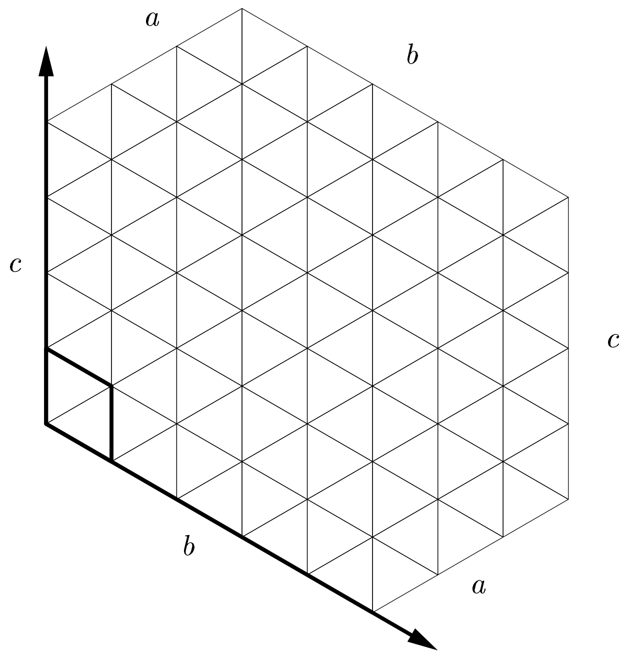
A rhombus tiling of a hexagon with an indication
of the corresponding family of non-intersecting lattice paths.

FIGURE 5



The family of non-intersecting lattice paths corresponding to the rhombus tiling in Figure 5.

FIGURE 6



The oblique angled coordinate system.

FIGURE 7