Semidefinite Programs and Association Schemes

Michel X. Goemans*
CORE
B-1348 Louvain-La-Neuve, Belgium
goemans@math.mit.edu

Franz Rendl†
Universität Klagenfurt
Institut für Mathematik
A-9020 Klagenfurt, Austria
franz.rendl@uni-klu.ac.at

July 1999

Abstract

We consider semidefinite programs, where all the matrices defining the problem commute. We show that in this case the semidefinite program can be solved through an ordinary linear program. As an application, we consider the max-cut problem, where the underlying graph arises from an association scheme.

Key words: semidefinite programming, association scheme, maximum cut problem

1 Introduction

Semidefinite programs (SDP) have recently turned out to be a powerful tool in many areas of applied mathematics, notably in combinatorial optimization. We consider the following 'primal' form of SDP. For given symmetric matrices $M_j, j = 0, \ldots, m$ of order n and a vector b find

$$\text{(SDP-P)} \quad \max\langle M_0, X\rangle \ \text{ such that } \langle M_j, X\rangle = b_j,\ j = 1, \ldots, m \text{ and } X \succeq 0,$$

where $\langle A, B\rangle$ denotes the Frobenius inner product tr $AB = \sum_{i,j} A_{ij}B_{ij}$, and $A \succeq 0$ imposes the positive semidefiniteness of A. This is a convex optimization problem. Its dual can be defined to be

$$\text{(SDP-D)} \quad \min b^T y \text{ such that } \sum_{j\geq 1} y_j M_j - M_0 =: Z \succeq 0, \text{ and } y \text{ unconstrained.}$$

The formulation (SDP-P) contains only equality constraints (in addition to the semidefinite constraints). Inequality constraints can also be included with the obvious modification that a dual variable y_j to an inequality constraint $\langle M_j, X\rangle \leq b_j$ must be nonnegative.

The duality relations between (SDP-P) and (SDP-D) are more subtle than linear programming duality. Weak duality, stating that

$$0 \leq b^T y - \langle M_0, X\rangle = \langle X, Z\rangle$$

*On leave from MIT. Research partially supported by the Training and Mobility of Researchers Programme DONET (contract number ERB FMRX-CT98-0202) and NSF contract 9623859-CCR.

†Financial support from the FWF Project P12660-MAT is gratefully acknowledged.

for feasible X, y, Z is easy to see. To insure strong duality, or equivalently the existence of a feasible X and Z such that $\langle X, Z\rangle = 0$, it is customary to assume some *constraint qualification* to hold. For example, strong duality holds whenever the primal and the dual have strictly feasible solutions (i.e. solutions for which X and Z are positive definite).

From a computational point of view, these types of problems can be solved efficiently (to some prescribed precision) using interior point methods, see [14]. Recent surveys showing the connection between SDP and combinatorial optimization can be found for instance in [5, 16].

In this note we investigate SDPs where the matrices defining the problem arise from an *association scheme*. We review their definition and basic properties in the next section. Association schemes are much studied in algebraic graph theory and in coding theory. The dissertation of Delsarte [4] was a fundamental step in their study. For coding theory purposes, he was interested in deriving bounds on the cardinality of the maximum stable set (a stable set is a set of mutually non-adjacent vertices) for graphs whose incidence matrix arises from an association scheme. He introduced and developed a beautiful linear programming bound which exploits the strong properties of association schemes. A few years later, in his study of the Shannon capacity, Lovász [9] proposed an upper bound (the so-called theta function) on the size of any stable set for an arbitrary graph. This theta function can be expressed in many different ways, including as an SDP. Schrijver [17] then showed that, for association schemes, a refinement of the theta function (with nonnegativity constraints added to one of the formulations) is equivalent to the LP-bound of Delsarte. In the context of the maximum cut problem, Karloff [7] and Alon and Sudakov [1] have also exploited simplifications of SDP for instances that arise from association schemes.

In this note we will derive the following two results.

- SDP over an association scheme, or more generally, SDP where the input matrices commute, are equivalent to ordinary LP (Section 3).
- SDP over an association scheme allow constraint aggregation for certain types of constraints. This will be elaborated upon for SDP relaxations of Max-Cut (Section 4). Roughly speaking, in some cases, we can replace a subset of constraints by a single constraint, and still guarantee that each individual constraint is satisfied.

Notation: We denote by J the matrix of all ones, e denotes the vector of all ones, $\mathrm{Diag}(v)$ is the diagonal matrix with vector v on its diagonal, the vector $\mathrm{diag}(X)$ contains the main diagonal of the matrix X.

2 Association Schemes

A set of $l+1$ symmetric 0,1-matrices $A_0, \ldots, A_l$ of order n forms a (symmetric) association scheme $\mathcal{A}$ if the following properties hold.

A1. $A_0 = I$,

A2. $\sum_{i=0}^{l} A_i = J$,

A3. there exist p_{ij}^k $(0 \le i, j, k \le l)$ such that $A_i A_j = \sum_{k=0}^{l} p_{ij}^k A_k$.

The vector space generated by the linear combinations (over the reals) of the matrices A_i's is known as the Bose-Mesner algebra of the scheme. This algebra is commutative because condition A3 implies that $A_i A_j$ is symmetric, hence equal to $A_j A_i$. For simplicity, we will denote by $\mathcal{A}$ the set of all linear combinations of the A_i's (as well as the association scheme itself); when we refer to a matrix M contained in an association scheme $\mathcal{A}$, it simply means that $M = \sum_i \chi_i A_i$ for some $\chi_i \in R$. The above definition of association schemes in fact considers only the *symmetric* case, a similar definition can be made without assuming that the matrices A_i be symmetric. We refer to [2] for further results on association schemes.

Association schemes are important structures in algebraic graph theory and in coding theory. The classes of *strongly regular graphs*, *distance regular graphs* and *symmetric circulants* are instances of association schemes. In coding theory, the most important association schemes are the *Hamming and Johnson schemes.* The Hamming scheme consists of $n+1$ matrices of order 2^n whose rows and columns are indexed by n-bit strings. An entry of matrix A_i is equal to 1 iff the corresponding bit strings differ in exactly i positions.

From condition A3 it follows that the A_i form a commuting family of matrices, and hence can be diagonalized simultaneously. In other words, they (as well as any matrix in $\mathcal{A}$) share a system of orthonormal eigenvectors. Furthermore, it can be shown that the Bose-Mesner algebra possesses a basis with very strong properties, see [2]. Namely, there exist symmetric matrices $E_0, \dots, E_l$ such that

E1. $E_i E_j = \delta_{ij} E_i$,

E2. $\sum_i E_i = I$,

E3. $E_0 = \frac{1}{n} J$,

E4. $\{E_0, \dots, E_l\}$ is a basis of $\mathcal{A}$.

Property E4 implies that each of the E_i can be expressed through the $A_j's$ and vice versa. Using standard notation for association schemes, we can thus write

$$A_j = \sum_{i=0}^{l} P_{ij} E_i. \tag{1}$$

The matrices E_i are idempotent ($E_i^2 = E_i$ by property E1), and thus have only 0 and 1 as eigenvalues, and hence are positive semidefinite. In fact, it can be shown that E_i can be expressed as $\sum_{k \in B_i} q_k q_k^T$ where the q_k's for $k \in B_i$ form an orthonormal basis of the ith common eigenspace of the A_j's. The fact that $E_i E_j = 0$ for $i \neq j$ and condition E2 imply that any eigenvector v of any of the E_j's corresponds to the eigenvalue 1 for exactly one of the E_j's and to the eigenvalue 0 for all the others. Hence, the eigenvalues of $\sum_i \lambda_i E_i$ are the λ_i's (with appropriate multiplicities); there are thus at most $l+1$ distinct eigenvalues. We define $\mu_i := \text{tr } E_i$, and since E_i is idempotent $\mu_i = \text{rank}(E_i)$. Thus, by (1), the eigenvalues of A_j are the P_{ij}'s with multiplicity μ_i for $i = 0, \cdots, l$. Observe that, since E_i belongs to $\mathcal{A}$, its diagonal elements are constant and equal to $\frac{\mu_i}{n}$, i.e.

$$\text{diag}(E_i) = \frac{\mu_i}{n} e. \tag{2}$$

3 Exploiting Commutativity

Let us suppose now that the input matrices $\{M_0, \dots, M_m\}$ for SDP form a commuting family of matrices, i.e. $M_i M_j = M_j M_i$ for all i, j. A well known fact, see e.g. ([11],Theorem 2.3.3), states that in this case there exists an orthogonal matrix $Q = (q_1, \dots, q_n)$, such that

$$Q^T M_j Q = \text{Diag}(a_j),$$

where $a_j = (a_{ji})$ is the (row)vector of eigenvalues of M_j. We collect the eigenvalues of $M_j,\ j \geq 1$ in the matrix $A = (a_{ij})$. Hence, using $E_i := q_i q_i^T$, we have

$$M_j = \sum_i a_{ji} q_i q_i^T = \sum_i a_{ji} E_i. \tag{3}$$

Observe furthermore that $E_i E_j = \delta_{ij} E_i$ and therefore, as in the case of association schemes, the eigenvalues of any matrix of the form $\sum_i \lambda_i E_i$ are the λ_i's. Finally, we assume that both (SDP-P) and (SDP-D) are feasible. No further constraint qualification is necessary.

The key observation is now the slack matrix Z of (SDP-D) must also be contained in the commutative algebra generated by the E_i's, because

$$Z = \sum_{j \geq 1} y_j M_j - M_0 = \sum_i (\sum_j y_j a_{ji} - c_i) E_i.$$

(Here we have set $c := a_0^T$, to follow LP notation.) Since the eigenvalues of Z are the entries of $A^T y - c$, we have that

$$Z \succeq 0 \text{ if and only if } A^T y \geq c.$$

In plain words, the semidefiniteness of Z can be expressed by a finite number of inequalities. Thus (SDP-D) is equivalent to the following linear program (LP-D)

$$\text{(LP-D)} \quad \min b^T y \text{ such that } z = A^T y - c \geq 0.$$

This problem is by assumption feasible. Feasibility of (SDP-P) shows that (SDP-D) is bounded from below because of weak duality. Hence, (LP-D) has a finite optimum, which is equal to the optimal value of its dual program (LP-P):

$$\text{(LP-P)} \quad \max c^T x \text{ such that } Ax = b,\ x \geq 0.$$

Suppose now that x is optimal for (LP-P) and y, z is optimal for (LP-D). Let us define

$$X := \sum_i x_i E_i \ \text{ and } Z := \sum_i z_i E_i. \tag{4}$$

Then clearly $X \succeq 0, Z \succeq 0$, because $x, z \geq 0$. By construction, Z is feasible for (SDP-D). Since

$$\langle M_j, X \rangle = \sum_i a_{ji} x_i = (Ax)_j = b_j,$$

the matrix X is also feasible for (SDP-P). Finally, we note that $\langle X, Z \rangle = x^T z = 0$, because of complementary slackness for x and z. As a consequence, we can solve (SDP-P) by solving the linear program (LP-P). Summarizing, we have proved the following result.

Theorem 1 *Suppose that the matrices defining (SDP-P) commute. Suppose further that both (SDP-P) and (SDP-D) are feasible. Then these problems have optimal solutions X and Z and these solutions can be computed through the linear programs (LP-P) and (LP-D) using (4). Strong duality holds without further constraint qualification.*

This theorem reduces SDP over a commuting family of matrices of order n to LP with n variables. The number m of constraints is unchanged. If, in addition, the matrices are contained in an association scheme, the columns of A (containing the eigenvalues of the M_j's) can be partitioned into $l+1$ groups of columns such that group k ($k = 0, \cdots, l$) contains μ_k identical columns. By keeping only one column for each group, we reduce the number of variables to $l+1$ which can be much smaller than n. For the Hamming scheme for instance, the initial SDP involves matrices of size $2^n \times 2^n$, while (LP-P) has only $n+1$ variables after suppressing identical columns. Since we keep only one column for each group (and thus aggregate over each eigenspace), we need to scale the primal variables by the multiplicity μ_i and use the E_i defined in Section 2. Therefore, the matrix A in (LP-P) and (LP-D) is now such that $M_j = \sum_i a_{ji} E_i$ (as in (3)). In summary, we derive the following:

Corollary 2 *Suppose that the matrices defining (SDP-P) are contained in an association scheme $\mathcal{A}$. Suppose further that both (SDP-P) and (SDP-D) are feasible. Then these problems have optimal solutions X and Z, which are also contained in $\mathcal{A}$. These solutions can be computed through the linear programs (LP-P) and (LP-D), where A is such that $M_j = \sum_i a_{ji} E_i$ and E_i is defined in Section 2, using*

$$X := \sum_i \frac{x_i}{\mu_i} E_i \text{ and } Z := \sum_i z_i E_i. \tag{5}$$

This type of simplification of SDP was already used by Schrijver [17] to show the equivalence between a stronger variant of the theta function of Lovász and the LP bound of Delsarte for graphs $G = (V, E)$ arising from association schemes. We should point out that if we only assume that the incidence matrix A commutes with J (or with $J - A$), i.e. that the graph is regular, then the simplification derived by Schrijver [17] does not apply, see [5] for details. This can be explained by the fact that an aggregation of constraints is needed to transform the semidefinite program defining the theta function (or its stronger variant) into the right form; it is only in the case of association schemes that this aggregation can be performed without changing the optimum value of the program.

In the next section we explore this contraint aggregation issue in the case of the maximum cut problem arising from association schemes.

4 Constraint Aggregation and the Maximum Cut Problem

The following quadratic problem in -1,1 variables plays a central role in combinatorial optimization:

$$z_{mc} := \max\{x^T L x : \ x \in \{-1, 1\}^n\}.$$

The matrix L is without loss of generality assumed to be symmetric. This problem is equivalent to the *maximum cut problem* or MAX-CUT, which asks to partition the vertices of an undirected edge-weighted graph $G = (V, E)$ with edge weights $w_{ij} : (i, j) \in E$ into two sets so as to maximize the total weight of edges joining the two sets. This problem is well-known to be NP-complete [12]. To model this problem formally, we represent partitions (S, T) of V by -1,1 vectors x with $x_i = 1$ exactly if $i \in S$. A vector x with all $x_i \in \{-1, 1\}$ is called a cut vector. If we assume that $w_{ij} = 0$ for $(i, j) \notin E$, then the MAX-CUT problem is characterized by the weight matrix $W = (w_{ij})$. We denote by $L := \mathrm{Diag}(We) - W$ the Laplacian of the matrix W. The weight of the partition, given by the cut vector x, can easily be shown to be $\frac{1}{4} x^T L x$. For notational convenience, we drop the factor $\frac{1}{4}$ from this cost function.

Rewriting $x^T L x = \langle L, x x^T \rangle$, we get the following well-known semidefinite relaxations for MAX-CUT, see [13, 3, 6, 10]:

$$z_{mc} \ \leq \ z_{psd-met} := \max\{\langle L, X \rangle : \mathrm{diag}(X) = e, X \succeq 0, X \in \mathrm{MET}\} \tag{6}$$

$$\leq \quad z_{psd} := \max\{\langle L, X\rangle : \operatorname{diag}(X) = e, X \succeq 0\} \tag{7}$$
$$\leq \quad z_{mp} := \max\{\langle L, X\rangle : \operatorname{tr}(X) = n, X \succeq 0\} = n\lambda_{\max}(L). \tag{8}$$

The bound z_{mp}, introduced in [13], is in general much weaker than z_{psd}. The relaxation $z_{psd-met}$ is obtained by tightening z_{psd} to include all *triangle inequalities.* By definition, $X \in$ MET precisely if $x_{ij} + x_{jk} + x_{ik} \geq -1, x_{ij} - x_{ik} - x_{jk} \geq -1$ holds for all distinct triples (i, j, k). This relaxation $z_{psd-met}$ is still tractable because there are only polynomially many, namely $4\binom{n}{3}$, constraints defining MET. Currently, this relaxation is still a computational challenge, and practical implementations have serious difficulties for $n \approx 100$.

The relaxation z_{psd} has atttracted a lot of attention due to the work of Goemans and Williamson. They have shown [6] that this semidefinite program approximates z_{mc} quite closely. They give a randomized algorithm producing a cut whose expected value $E[cut]$ satisfies

$$\frac{E[cut]}{z_{psd}} \geq \alpha,$$

where $\alpha = \frac{2}{\pi}\min_{0<\theta\leq\pi}\frac{\theta}{1-\cos\theta} \geq 0.87856$ whenever $W \geq 0$.This implies that $z_{mc} \geq \alpha z_{psd}$. Although it is not known whether $\frac{z_{mc}}{z_{psd}}$ can be arbitrarily close to α, Karloff [7] has proved that, for graphs arising from the Johnson (association) scheme, $\frac{E[cut]}{z_{psd}}$ can be arbitrarily close to α. In [6], it was shown that the performance guarantee of α can be improved whenever the maximum cut is known to be a large fraction of the total weight of the edges (see [6] for a precise statement). Alon and Sudakov [1] generalized Karloff's result by showing that the improved performance guarantee for $\frac{E[cut]}{z_{psd}}$ shown in [6] can also be arbitrarily closely approached, and in this case the instances they consider arise from the Hamming scheme. Both in [7] and in [1], the following phenomenon happens. Although they exhibit an optimum solution X^* to the semidefinite programming relaxation leading to the poor behavior, they also show that $z_{psd} = z_{mc}$ and in fact X^* can be expressed as a convex combination of optimum cut matrices.

This suggests to consider instances where L is contained in some association scheme. Thus let us assume now that the cost matrix W of the underlying Max-Cut problem comes from an association scheme $\mathcal{A}$. Poljak and Turzik show that Max-Cut on a very restricted subset of symmetric circulants is tractable [15], but its complexity status for association schemes in general is open. Since $L = \operatorname{Diag}(We) - W$ and e is an eigenvector of W (independently of the association scheme being considered, since it is an eigenvector of J), we can express the Laplacian L as a linear combination of the A_j's, and get

$$L = \sum_j \alpha_j A_j = \sum_i (\sum_j \alpha_j P_{ij}) E_i = \sum_i \lambda_i E_i. \tag{9}$$

The numbers $\lambda_i := \sum_j \alpha_j P_{ij}$ are the eigenvalues of the Laplacian L.

We note first that the relaxation z_{mp} can be viewed under the framework of SDP over commuting matrices, because I commutes with any L. Furthermore, we get from Corollary 2 that the optimum solution X can be expressed as

$$X = \frac{n}{\mu_p} E_p, \tag{10}$$

where $p = \arg\max_i \lambda_i$. In fact, we get more from this solution. Because of (2), note that X from (10) satisfies $\operatorname{diag}(X) = e$. Therefore X is also an optimal solution to the second relaxation z_{psd}, and we have that $z_{mp} = z_{psd}$ for Laplacians arising from association schemes. This would not be surprising if all the constraints $x_{ii} = 1$ of $\text{diag}(X) = e$ were contained in $\mathcal{A}$. But clearly they are not, unless $n = 1$. But we were able to aggregate them into the single equation $\langle I, X\rangle = n$ and apply Corollary 2.

The flexibility of choosing a positive semidefinite matrix generated by elements in the pth eigenspace was already exploited in [7, 1]. Matrices forming an association scheme have quite a strong combinatorial structure. This structure can be exploited to find sufficient conditions for $z_{mc} = z_{psd}$ to hold. Alon and Sudakov [1] investigate this in the case of the Hamming scheme. Since the eigenvectors of the A_i's can all be chosen to be $-1, +1$ vectors, such an eigenvector v for the eigenvalue λ_p leads to an optimum rank 1 matrix $X = vv^T$ for the SDP relaxation. In other words, for any MAX-CUT problem arising from the Hamming scheme, we have $z_{mc} = z_{psd}$. For the Johnson scheme, however, the situation is different. Not every eigenvalue (e.g., the even ones) has an eigenvector with all entries ± 1 (although there is a basis of $0, \pm 1$ eigenvectors, see [8]) and, as a result, one cannot guarantee that $z_{mc} = z_{psd}$ (and this fails even for K_3).

We conclude by showing how the triangle inequalities can be aggregated so that Corollary 2 can be applied. A single triangle inequality is not contained in any association scheme. Therefore we have to find subsets of these constraints, which can be added to form a matrix from $\mathcal{A}$.

Consider a triangle inequality, say $x_{ab} + x_{bc} + x_{ac} \geq -1$ where $(A_i)_{ab} = 1$, $(A_j)_{bc} = 1$ and $(A_k)_{ac} = 1$ (the indices i, j, k are not necessarily distinct). Consider now all the triangles having the same form, namely sets $\{a', b', c'\}$ such that $(A_i)_{a'b'} = 1$, $(A_j)_{b'c'} = 1$ and $(A_k)_{a'c'} = 1$. By aggregating all these triangle inequalities and using condition A3 (which says that the number of such triangles containing a given edge (a, c) with $(A_k)_{ac} = 1$ is p_{ij}^k), we get the inequality

$$p_{jk}^i \langle A_i, X\rangle + p_{ik}^j \langle A_j, X\rangle + p_{ij}^k \langle A_k, X\rangle \geq -a, \tag{11}$$

where $a = p_{ij}^k \langle A_k, J\rangle = p_{jk}^i \langle A_i, J\rangle = p_{ki}^j \langle A_j, J\rangle$. We were thus able to aggregate triangle inequalities into inequalities induced by the association scheme. On the other hand, consider $X \in \mathcal{A}$, i.e. $X = \sum_r \xi_r A_r$ for given reals ξ_r. Since

$$x_{ab} + x_{bc} + x_{ac} = \sum_r \xi_r \left[(A_r)_{ab} + (A_r)_{bc} + (A_r)_{ac}\right] = \xi_i + \xi_j + \xi_k,$$

for all triangles of the same form, we have that (11) implies that $x_{ab} + x_{bc} + x_{ac} \geq -1$ for any such triangle. This means that we have not weakened the relaxation by aggregating the triangle inequalities of the same form into a single constraint. As a consequence of Corollary 2, the resulting semidefinite relaxation, with all triangle inequalities included, can be solved as an ordinary linear program. Since the aggregation will replace multiples of n constraints by a single constraint, the resulting LP will have no more than $O(n^2)$ inequality constraints, instead of $O(n^3)$ constraints in the original SDP relaxation. The number of resulting inequalities can also be bounded by $O(l^3)$ which can be much smaller as in the case of the Hamming scheme. Computational experiments with this relaxation for circulants L were carried out by the authors, and problems of size $n \approx 1000$ could be handled easily.

If we were trying to follow the same approach for more complex inequalities, such as the 5-cycle inequalities saying that $x_{ab} + x_{bc} + x_{cd} + x_{de} + x_{ea} \geq -3$, we would run into problems since it is not clear how to aggregate them without taking into account the specific association scheme being considered. For certain association schemes, such as the one inducing circulant graphs, this can be done (by exploiting symmetry).

Finally, it is interesting to note that bisection problems with cardinality constraints on the partitions S and T also fall into this framework. To see this, note that $|S| = k$ translates into $e^T x = 2k - n$. Therefore, we can model $|S| = k$ by $\langle J, X \rangle = (2k-n)^2$. The matrix J is again contained in any association scheme.

References

[1] N. ALON and B. SUDAKOV. Bipartite subgraphs and the smallest eigenvalue. Technical Report, 1998.

[2] A.E. BROUWER and W.H. HAEMERS. Association schemes. In *Handbook of Combinatorics*, R.L. Graham, M. Grötschel, and L. Lovász, eds., Elsevier, 747–771, 1995.

[3] Ch. DELORME and S. POLJAK. Laplacian eigenvalues and the max-cut problem. *Mathematical Programming*, 63:557–574, 1993.

[4] P. DELSARTE. An algebraic approach to the association schemes of coding theory. Philips Research Reports, Supplement, 10, 1973.

[5] M.X. GOEMANS. Semidefinite programming in combinatorial optimization. *Mathematical Programming*, 79:143–161, 1997.

[6] M.X. GOEMANS and D.P. WILLIAMSON. Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming. *Journal of the ACM*, 42: 1115–1145, 1995.

[7] H. KARLOFF. How good is the Goemans-Williamson MAX CUT algorithm. In *Proc. of the 28th ACM Symp. on Theory Comput.*, pages 427–434, 1996.

[8] D.E. KNUTH. Combinatorial Matrices. Notes written for the Institut Mittag-Leffler (1991). Revised version (from 1996) available at `http://www-cs-faculty.stanford.edu/~knuth/papers/cm.tex.gz`.

[9] L. LOVÁSZ. On the Shannon capacity of a graph. *IEEE Transactions on Information Theory*, IT-25:1–7, 1979.

[10] C. HELMBERG, F. RENDL, R. J. VANDERBEI, and H. WOLKOWICZ. An interior point method for Semidefinite Programming. *SIAM Journal on Optimization* 6:342–361, 1996.

[11] R.A. HORN and C.R. JOHNSON. Matrix Analysis. Cambridge University Press 1985.

[12] R. M. KARP. Reducibility among combinatorial problems. In R. E. Miller, J. W. Thather, editors, *Complexity of Computer Computation*, 85–103, Plenum Press, New York, 1972.

[13] B. MOHAR, and S. POLJAK. Eigenvalues and the Max-Cut Problem. *Czech. Math. J.* 115:343–352, 1990.

[14] Y. NESTEROV and A. NEMIROVSKII. *Interior–Point Polynomial Methods in Convex Programming.* SIAM Studies in Applied Mathematics, Philadelphia, 1994.

[15] S. POLJAK, and D. TURZÍK. Max-Cut in circulant graphs. *Discrete Mathematics* 108: 379–392, 1992.

[16] F. RENDL. Semidefinite programming and combinatorial optimization. *Applied Numerical Mathematics*, 29:255–281, 1999.

[17] A. SCHRIJVER. A comparison of the Delsarte and Lovász bounds. *IEEE Transactions on Information Theory*, 25:425–429, 1979.